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**T. Hayashi**

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*M. Fujiwara,* *J. Ishiwara,*
*T. Kubota,* *S. Kakeya,* *T. Kojima.*

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THE TÔHOKU MATHEMATICAL JOURNAL

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# On a Determinate System of Non-independent Trials,

by

MAGOICHIRÔ WATANABE, Tôkyô.

Contents.

Introduction.
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Part II. System of Trials of the generalized zero Class with respect to an Event.
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Part V. Determinate System of Trials with respect to a System of mutually exclusive Events.
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## Introduction.

If $s$ denote the number of the happening of an event in $n$ trials, non-independent in general, the probability of the event happening in any one of the trials being capable of varying depending on the occurrence of the event in other trials, and $\pi_n(\alpha, \beta)$, the probability that $s$ will satisfy the inequality

$$\alpha \leqq \frac{s}{n} \leqq \beta,$$

where $\alpha$ and $\beta$ are numbers between 0 and 1, both limits inclusive, then the limit $\pi(\alpha, \beta)$, in case when it exists, to which $\pi_n(\alpha, \beta)$ tends when $n$ becomes infinite, is a certain function of $\alpha$ and $\beta$. Now a system of trials may be such that $\pi(0, x)$ exists at least at points $x$ forming an everywhere dense set of points in the interval $(0 \leqq x \leqq 1)$(1). Such

(1) In this case $\pi(0, x)$ is a monotonic function of $x$ in the domain consisting of points of the set, and therefore if $\bar{\lambda}(x)$ denote the extended function of $\pi(0, x)$, then at points of continuity of $\bar{\lambda}(x)$ which form a set of the measure (in Borel's sense) equal to the length of the interval $(0, 1)$, we have easily

a system of trials is called the *determinate system with respect to the event under consideration.* The investigation of the conditions for the occurrence of a system of trials for which this is the case, constitutes the subject of the first three Parts of the present paper. I shall consider in the Part I an important class which is characterized by a very special condition and in the Part II another special class, while in the Part III the most general one including the two former classes as its special ones.

Again if $s_1, s_2, \cdots\cdots, s$ denote the numbers of the happening of $l$ mutually exclusive events in $n$ trials respectively such that

$$s_1+s_2+\cdots\cdots+s_l \leqq n,$$

and $\pi_n(\alpha, \beta)$, the probability that $s_1, s_2, \cdots\cdots, s_l$ will satisfy simultaneously

$$\alpha_i \leqq \frac{s_i}{n} \leqq \beta, \quad (i=1, 2, \cdots\cdots, l)$$

where $\alpha(\alpha_1, \alpha_2, \cdots\cdots, \alpha_l)$ and $\beta(\beta_1, \beta_2, \cdots\cdots, \beta_l)$ are the points in the interior or on the boundary of the closed $l$-dimensional domain $\overline{D}$ determined by

$$0 \leqq t \leqq 1, \qquad (i=1, 2, \cdots\cdots, l)(^1),$$

then the limit $\pi(\alpha, \beta)$ to which $\pi_n(\alpha, \beta)$ tends when $n$ becomes infinite is a certain function of $\alpha$ and $\beta$. Now a system of trials may be such that $\pi(0, t)$ exists at least at points $t$ forming an everywhere dense set of points in $\overline{D}$ ($^2$). Such a system of trials is called the *determinate system with respect to a given system of $l$ mutually exclusive events.* The two Parts IV and V will be devoted to the discussion of a system of trials for which this is the case.

The results obtainable for the system of trials just considered may admit of easy generalization which corresponds to Tchebycheff's ex-

---

$$\overline{\lambda}(x)=\pi(0, x),$$

so that the existence of $\pi(0, x)$ at an everywhere dense set of points in $(0, 1)$ implies at the same time its continuity at points of a set also everywhere dense in the interval $(0, 1)$.

($^1$) That the probability that the point $\left(\frac{s_1}{n}, \frac{s_2}{n}, \cdots, \frac{s_l}{n}\right)$ lies outside the domain $D$ where $D$ is defined by $0 \leqq t_i \leqq 1$ $(i=1, 2, \cdots, l)$, $t_1+t_2+\cdots\cdots+t_l \leqq 1$, vanishes is to be taken into account.

($^2$) In this case, $\pi(0, t)$ is a continuous function of $t$ in a domain consisting of points of a set whose measure is equal to that of the domain $\overline{D}$.

tension of Poisson's Law of Large Numbers. A short discussion of such a generalization will be given in the last Part of the present paper.

## Part I.

## System of Trials of the first Class with respect to an Event.

**1.** Let $s$ denote the number of the happening of an event in $n$ trials, non-independent in general, the probability of the event in $m$th trial being $p_m$ *à priori* but capable of varying depending on the occurrence of the event in other trials; and let $\pi_n(x;\varepsilon)$ denote the probability that $s$ will satisfy the inequality

$$\left|\frac{s}{n}-x\right|\leqq\varepsilon\text{(}^1\text{)},$$

where $x$ is a number between 0 and 1 both limits inclusive and $\varepsilon$, a given positive number. Then the limit $\pi(x;\varepsilon)$, if exist, to which $\pi_n(x;\varepsilon)$ tends when $n$ becomes infinite, is a certain function of $x$ and $\varepsilon$.

A point $x$ within the closed interval $(0,1)$ may be such that, by taking a sufficiently small positive number $\varepsilon$,

$$\pi(x;\varepsilon)=0.$$

Let us call such a point $x$ the *point of the class* 0 and let the set of such points be denoted by $A_0$([2])([3]).

Again a point $x$, not belonging to the set $A_0$, may be such that, for all positive $\varepsilon$ smaller than a fixed positive number $\varepsilon_1$,

$$\pi(x;\varepsilon)=\lambda,$$

where $\lambda$ is a definite positive number. Let us call such a point $x$ the

---

(1) That the probability of the existence of $\frac{s}{n}>1$ or $\frac{s}{n}<0$ always vanishes is to be taken into account.

(2) If we extend this definition to external points of the interval $(0,1)$, it is evident that all such points belong to this set $A_0$. On this reason we shall frequently include points external to the interval $(0,1)$ as belonging to the set $A_0$.

(3) The point of the class 0 may of course be defined by $\pi(x-\varepsilon,\ x+\varepsilon')=0$ and the point of the class 1, by $\pi(x-\varepsilon,\ x+\varepsilon')=\lambda\neq 0$. It will be seen later that the point of another character may conveniently be defined by means of $\pi(x-\varepsilon,\ x+\varepsilon')$.

*point of the class* 1 and let the set of such points be denoted by $A_1$(1).

Now it can be shown that

*If all points within a closed interval* $(\alpha, \beta)$ *belong to the set* $A_0$, *then the probability* $\pi_n(\alpha, \beta)$ *that* $s$ *will satisfy the inequality*

$$\alpha \leqq \frac{s}{n} \leqq \beta,$$

*also approaches zero as limit when* $n$ *becomes infinite.*

For suppose
$$\lim_{n=\infty} \pi_n(\alpha, \beta) \neq 0. \tag{1}$$

Then if we consider the set of points $(x_1)$ for which the probability $\pi_n(\alpha, x_1)$ that $s$ will satisfy

$$\alpha \leqq \frac{s}{n} \leqq x_1,$$

approaches zero as limit when $n$ becomes infinite, the number $x_1$ which exists certainly as $\alpha$ belongs to the set $A_0$, must have an upper limit $k$ not greater than $\beta$. We have, for the point $x=k$, by taking a sufficiently small positive number $\varepsilon_1$,

$$\lim_{n=\infty} \pi_n(k\,;\ \varepsilon_1)=0,$$

so that

$$\lim_{n=\infty}\left\{\pi_n(\alpha, k-\varepsilon_1)+\pi_n(k-\varepsilon_1,\ k+\varepsilon_1)\right\}=0.$$

This shows that the probability that $s$ will satisfy

$$\alpha \leqq \frac{s}{n} \leqq k+\varepsilon_1,$$

approaches zero when $n=\infty$ which contradicts the hypothesis that $k$ is the upper limit of the set $(x_1)$. The assumption (1) is inadmissible and therefore

$$\lim_{n=\infty} \pi_n(\alpha, \beta)=0.$$

From this it follows immediately that *the set* $A_0$ *can not consist of*

---

(1) In a system of independent trials or a system of non-independent trials of a special class treated by me in the Tôhoku Math. Journ. Vol. 12 (1917) pp. 24–42, the set $A_1$ consists of a single point $x=p$ and the set $A_0$ consists of all other points in the continuum (0, 1) if $\lim_{n=\infty} \frac{1}{n} \sum_1^n p_m$ exist and have the value $p$.

*all points of the closed interval* (0, 1), *i. e. there must be at least a point not belonging to the set* $A_0$.

For if the set $A_0$ consists of all points of the closed interval (0, 1) the probability that $s$ will satisfy

$$0 \leqq \frac{s}{n} \leqq 1,$$

which is clearly equal to 1, will vanish at the limit when $n=\infty$ which is absurd.

Furthermore we may show that

*The set* $C(A_0)$, *complementary to* $A_0$ *in the closed interval* (0, 1), *which has just been proved always to exist, is a closed set.*

For if $x_1$ be a point of the set $A_0$ at which

$$\pi(x_1; \varepsilon)=0,$$

then all the points $x$ lying between $x_1-\varepsilon$ and $x_1+\varepsilon$ and also belonging to the interval (0, 1) belong to the same set and therefore the point $x_1$ can not be a limiting point of its complementary set, so that any limiting point of $C(A_0)$ always belongs to the set $C(A_0)$.

This result may also be expressed as follows:

*The set* $A_0$ *consists of points in the interior of non-overlapping intervals with or without the end points* $x=0$ *and* $x=1$.

As regards the set $A_1$, it is a component of $C(A_0)$ and all the points belonging to the set are isolated, *i.e.* the set is itself isolated. Its derived set therefore consists of points which belong neither to $A_0$ nor to $A_1$.

Consequently *in the case when there exists no point in the interval* (0, 1) *which belongs neither to* $A_0$ *nor to* $A_1$, *the set* $A_1$ *must consist of at least one point and at most a finite number of points.*

Such a system of trials is said to be of the *first class* with respect to the event under consideration.

The occurrence of such a case requires of course some conditions to be satisfied which will be fully discussed in the following articles.

**2.** We shall first consider necessary conditions for the existence of such a case.

Let $x_1, x_2, \cdots\cdots, x_r$ be the points of the set $A_1$, all other points in the interval (0, 1)([1]) belonging to the set $A_0$; and let $x_i$ be the point

([1]) Here and hereafter the interval (0, 1) will be understood in the sense of "closed interval."

at which

$$\pi(x_i;\ \varepsilon)=\lambda_i, \qquad (i=1, 2, \cdots\cdots, r)$$

$\varepsilon_i$ being taken sufficiently small.

Again let $F(n, s)$ be the probability of the happening of the event under consideration just $s$ times in $n$ trials. If we consider the expression

$$T_m(n)=\sum_{s=0}^{n}\left(\frac{s}{n}\right)^m F(n, s), \qquad m\leqq 0, \tag{2}$$

then it can be written in the form

$$T_m(n)=\sum_1\left(\frac{s}{n}\right)^m F(n, s)+\sum_2\left(\frac{s}{n}\right)^m F(n, s)+\cdots\cdots+\sum_r\left(\frac{s}{n}\right)^m F(n, s)$$
$$+\sum{}'\left(\frac{s}{n}\right)^m F(n, s),$$

where $\sum_i$ denotes the summation about $s$ which satisfies

$$\left|\frac{s}{n}-x_i\right|\leqq\varepsilon_i, \qquad i=1, 2, \cdots\cdots, r,$$

$\varepsilon_i$ being a positive number arbitrarily chosen so small that within the interval $(x_i-\varepsilon_i,\ x_i+\varepsilon_i)$ there lies no point of the set $A_1$ except $x_i$, and $\sum'$ denotes the summation about $s$ complementary to $\sum_1+\sum_2+\cdots\cdots+\sum_r$.

Now we have

$$\sum{}'\left(\frac{s}{n}\right)^m F(n, s)\leqq\sum{}' F(n, s),$$

of which the right-hand side expression tends to zero when $n=\infty$ [§ 1]. And therefore

$$\lim_{n=\infty}\sum{}'\left(\frac{s}{n}\right)^m F(n, s)=0. \tag{3}$$

Again we have

$$(x_i-\varepsilon_i)^m\sum_i F(n, s)\leqq\sum_i\left(\frac{s}{n}\right)^m F(n, s)\leqq(x_i+\varepsilon_i)^m\sum_i F(n, s)$$

of which the left-hand side expression tends to $(x_i-\varepsilon_i)^m\lambda_i$ and the right-hand side expression tends to $(x_i+\varepsilon_i)^m\lambda_i$ when $n$ becomes infinite so that it gives, combined with (3),

$$\overline{\lim_{n=\infty}}\, T_m(n) \leqq \sum_{i=1}^{r} (x_i+\varepsilon_i)^m \lambda_i,$$

$$\underline{\lim_{n=\infty}}\, T_m(n) \geqq \sum_{i=1}^{r} (x_i-\varepsilon_i)^m \lambda_i.$$

Since these two relations hold for $\varepsilon_1, \varepsilon_2, \cdots\cdots, \varepsilon_r$, arbitrarily small, from which the left-hand side expressions are quite independent, it follows that $\lim_{n=\infty} T_m(n)$ should exist and

$$\lim_{n=\infty} T_m(n) = \sum_{i=1}^{r} x_i^m \lambda_i.$$

Putting

$$\lim_{n=\infty} T_m(n) = T_m, \tag{4}$$

we thus obtain *as necessary conditions for the existence of* $x_1, x_2, \cdots\cdots, x_r$ *of the set* $A_1$ *having their corresponding probabilities* $\lambda_1, \lambda_2, \cdots\cdots, \lambda_r$,

$$T_m = x_1^m \lambda_1 + x_2^m \lambda_2 + \cdots\cdots + x_r^m \lambda_r \text{ (1)}, \qquad m \geqq 0. \tag{5}$$

**3.** We shall next consider sufficient conditions for the existence of such a case.

*Suppose there exists a set of* $x_1, x_2, \cdots\cdots, x_r$; $\lambda_1, \lambda_2, \cdots\cdots, \lambda_r$ *which satisfy the* $2r+1$ *equations*

$$\left.\begin{aligned} 1 &= \lambda_1 + \lambda_2 + \cdots\cdots + \lambda_r, \\ T_1 &= x_1 \lambda_1 + x_2 \lambda_2 + \cdots\cdots + x_r \lambda_r, \\ T_2 &= x_1^2 \lambda_1 + x_2^2 \lambda_2 + \cdots\cdots + x_r^2 \lambda_r, \\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ T_{2r} &= x_1^{2r} \lambda_1 + x_2^{2r} \lambda_2 + \cdots\cdots + x_r^{2r} \lambda_r, \end{aligned}\right\} \tag{6}$$

*with the conditions*

$$\left.\begin{aligned} 0 \leqq x_i \leqq 1, \qquad x_i \neq x_k, \qquad (i \neq k) \\ \lambda_i \neq 0, \end{aligned}\right\} \tag{7}$$

and let us first show that

(i) *all the points excepting* $x_1, x_2, \cdots\cdots, x_r$ *in the interval* $(0, 1)$ *belong to the set* $A_0$,

(ii) *the point* $x_i$ $(i=1, 2, \cdots\cdots, r)$ *belongs to the set* $A_1$ *having its*

(1) It has been first pointed out by Prof. Kakeya that if the set $A_1$ consists of a single point, that point must be $x=p$. This can also be deduced from (5) as a special case if we observe that $Tp=1$.

*probability* $\pi(x_i; \varepsilon_i) = \lambda_i$.

To prove the first part we consider the non-negative expression $E(n)$ defined by

$$E(n) = \sum_{s=0}^{n} f\left(\frac{s}{n}\right) F(n, s), \tag{8}$$

where
$$f(x) = (x - x_1)^2 (x - x_2)^2 \cdots\cdots (x - x_r)^2. \tag{9}$$

If we write the polynomial $f(x)$ in the form

$$f(x) = x^{2r} + a_1 x^{2r-1} + \cdots\cdots + a_{2r},$$

we have

$$E(n) = \sum_{s=0}^{n} \left(\frac{s}{n}\right)^{2r} F(n, s) + a_1 \sum_{s=0}^{n} \left(\frac{s}{n}\right)^{2r-1} F(n, s) + \cdots\cdots + a_{2r} \sum_{s=0}^{n} F(n, s),$$

and hence, if $x_1, x_2, \cdots\cdots, x_r$; $\lambda_1, \lambda_2, \cdots\cdots, \lambda_r$ satisfy the relations (6),

$$\begin{aligned} \lim_{n=\infty} E(n) &= T_{2r} + a_1 T_{2r-1} + \cdots\cdots + a_{2r} \\ &= \sum_{i=1}^{r} \left\{ x_i^{2r} + a_1 x_i^{2r-1} + \cdots\cdots + a_{2r} \right\} \lambda_i \\ &= \sum_{i=1}^{r} f(x_i) . \lambda_i . \end{aligned}$$

Since $x_i$ is a zero point of the polynomial $f(x)$, it follows that

$$\lim_{n=\infty} E(n) = 0. \tag{10}$$

On the other hand if we write

$$\begin{aligned} E(n) = \sum_{1} f\left(\frac{s}{n}\right) F(n, s) + \sum_{2} f\left(\frac{s}{n}\right) F(n, s) + \cdots\cdots + \sum_{r} f\left(\frac{s}{n}\right) F(n, s) \\ + \sum{}' f\left(\frac{s}{n}\right) F(n, s), \end{aligned} \tag{11}$$

where $\sum_{i}$ and $\sum{}'$ have the same meaning as in the previous article, then for all $n$ and for $s$ belonging to the summation in $\sum{}'$, $f\left(\frac{s}{n}\right)$ has a positive lower limit $l$ different from zero if $\varepsilon_1, \varepsilon_2, \cdots\cdots, \varepsilon_r$ be fixed numbers however small. Hence we have

$$\begin{aligned} E(n) &\geqq \sum{}' f\left(\frac{s}{n}\right) F(n, s) \\ &\geqq l \sum{}' F(n, s), \end{aligned}$$

and therefore

$$\lim_{n=\infty} \Sigma' F(n, s) = 0. \tag{12}$$

Since $\varepsilon_1, \varepsilon_2, \cdots\cdots, \varepsilon_r$ may be taken arbitrarily small it follows that all the points except $x_1, x_2, \cdots\cdots, x_r$ in the interval $(0, 1)$ belong to the set $A_0$, which proves the first part of our proposition.

Again to prove the second part we consider the expression $E_i(n)$ defined by

$$E_i(n) = \sum_{s=0}^{n} f_i\left(\frac{s}{n}\right) F(n, s), \tag{13}$$

where

$$f_i(x) = \frac{A(x)}{(x - x_i)}, \qquad A(x) = (x - x_1)(x - x_2) \cdots\cdots (x - x_r). \tag{14}$$

For this expression we have

$$\begin{aligned} \lim_{n=\infty} E_i(n) &= \lambda_i \cdot f_i(x_i) \\ &= \lambda_i (x_i - x_1) \cdots\cdots (x_i - x_{i-1})(x_i - x_{i+1}) \cdots\cdots (x_i - x_r), \end{aligned} \tag{15}$$

which does not vanish, and by writing

$$\begin{aligned} E_i(n) = \sum_1 f_i\left(\frac{s}{n}\right) F(n, s) + \cdots\cdots + \sum_r f_i\left(\frac{s}{n}\right) F(n, s) \\ + \Sigma' f_i\left(\frac{s}{n}\right) F(n, s), \end{aligned} \tag{16}$$

it is easy to see by (12) that the last term $\Sigma' f_i\left(\frac{s}{n}\right) F(n, s)$ vanishes at the limit when $n = \infty$. Moreover it can be shown that

$$\lim_{n=\infty} \sum_j f\left(\frac{s}{n}\right) F(n, s) = 0, \qquad j \neq i. \tag{17}$$

For if $\varepsilon_j'$ be taken $(\varepsilon_j' \leqq \varepsilon_j)$, corresponding to an arbitrarily chosen positive number $\delta$, so as to ensure that

$$\left| f_i\left(\frac{s}{n}\right) \right| < \delta,$$

for all $s$ and $n$ satisfying $\left| \frac{s}{n} - x_j \right| \leqq \varepsilon_j'$, which is always possible if $j \neq i$, we have

$$\sum_j{}' \left| f_i\left(\frac{s}{n}\right)\right| F(n, s) < \delta \sum_j{}' F(n, s)$$
$$< \delta,$$

where $\sum_j{}'$ denotes the summation about $s$ satisfying $\left|\frac{s}{n} - x_j\right| \leqq \varepsilon_j'$, and therefore

$$\overline{\lim_{n=\infty}} \sum_j{}' \left| f_i\left(\frac{s}{n}\right)\right| F(n, s) \leqq \delta.$$

Since all the points within $|x - x_j| \leqq \varepsilon_j$ except $x = x_j$ belong to the set $A_0$ it follows that

$$\overline{\lim_{n=\infty}} \sum_j \left| f_i\left(\frac{s}{n}\right)\right| F(n, s) = \overline{\lim_{n=\infty}} \sum_j{}' \left| f_i\left(\frac{s}{n}\right)\right| F(n, s)$$
$$\leqq \delta,$$

a relation showing the truth of (17).

Consequently by the comparison of two relations (15) and (16) we get

$$\lim_{n=\infty} \sum_i f_i\left(\frac{s}{n}\right) F(n, s) = f_i(x_i) . \lambda_i. \tag{18}$$

This relation can only be satisfied when the point $x_i$ belongs to the set $A_1$ having its probability $\pi(x_i; \varepsilon_i) = \lambda_i$. For by denoting the upper and the lower limits of $f_i\left(\frac{s}{n}\right)$ for $\left|\frac{s}{n} - x_i\right| \leqq \varepsilon_i$ by $u$ and $l$ both of which do not vanish and have the same sign, we have

$$l \sum_i F(n, s) \leqq \sum_i f_i\left(\frac{s}{n}\right) F(n, s) \leqq u \sum_i F(n, s),$$

and therefore if $l$ and $u$ are positive

$$\frac{\sum_i f_i\left(\frac{s}{n}\right) F(n, s)}{u} \leqq \sum_i F(n, s) \leqq \frac{\sum_i f_i\left(\frac{s}{n}\right) F(n, s)}{l},$$

from which we obtain, by aid of (18),

$$\left.\begin{aligned} \overline{\lim_{n=\infty}} \sum_i F(n, s) &\leqq \frac{f_i(x_i)}{l} . \lambda_i, \\ \underline{\lim_{n=\infty}} \sum_i F(n, s) &\geqq \frac{f_i(x_i)}{u} . \lambda_i. \end{aligned}\right\} \tag{19}$$

The values of two expressions which stand on the right in (19) respectively may differ from $\lambda_i$ and therefore from each other by any assignable magnitude by taking $\varepsilon_i$ sufficiently small, while those on the left are quite independent from $\varepsilon_i$([1]) and hence we must have

$$\lim_{n=\infty} \sum_i F(n, s) = \lambda_i \text{([2])}.$$

Similar result holds when $l$ and $u$ are both negative. This proves the second part of our proposition.

**4.** The results obtained in the last two articles may be expressed as follows :—

*In order that a system of trials may be of the first class, with respect to an event, having $r$ points of the first class it is necessary and sufficient that the simultaneous equations* (6) *with the condition* (7) *are consistent.*

Now we shall show that *the condition* (7) *may be replaced by the mere condition*

$$\left.\begin{array}{ll} x_i \neq x_k, & (i \neq k) \\ \lambda_i \neq 0. & \end{array}\right\} \qquad (20)$$

In order to show this it will be sufficient to prove that if there exist a set of $x_1, x_2, \cdots\cdots, x_r$; $\lambda_1, \lambda_2, \cdots\cdots, \lambda_r$ satisfying (6) with the condition (20) then they satisfy the condition (7).

Now for a moment suppose that some of $x_1, x_2, \cdots\cdots, x_r$ satisfying (6) with the condition (20) do not lie in the interval (0, 1) or are imaginary. If among them $x_j$, $x_{j+1}, \cdots\cdots, x_{k-1}$, say, are real but do not lie in the interval (0, 1) and $x_k$, $x_{k+1}$; $x_{k+2}$, $x_{k+3}$; $\cdots\cdots$; $x_{l-2}$, $x_{l-1}$; say, are imaginary, of which $x_{k+2i}$ and $x_{k+2i+1}$ are conjugate to each other and $x_l$, $x_{l+1}, \cdots\cdots, x_r$, say, are isolated imaginaries([3]), then we take, in place of $f(x)$ considered in the last article, a non-negative polynomial $\bar{f}(x)$ where

---

([1]) $\sum_i F(n, s)$ depends of course on the choice of $\varepsilon_i$ in general; but its limit when $n=\infty$ does not depend at all on $\varepsilon_i$, for all the points within $|x-x_i| \leqq \varepsilon$ except $x_i$ belong to the set $A_0$.

([2]) It has been indirectly shown that, in case when the simultaneous equations (6) are consistent under the condition (7),

$$0 < \lambda_i \leqq 1 \qquad (i = 1, 2, \cdots, r).$$

([3]) This means that there is no pair of conjugate imaginaries among $x_l$, $x_{l+1}, \cdots$, $x_r$. We shall here assume their existence to arrive at a contradiction though their non-existence can directly be shown algebraically, as seen from the discussion in § 6.

$$\bar{f}(x)=(x-x_1)^2\cdots\cdots(x-x_{l-1})^2\{(x-x_l)(x-x_l')\}\cdots\cdots\{(x-x_r)(x-x_r')\},$$

$x_i'$ $(i=l, l+1,\cdots\cdots r)$ being the conjugate imaginaries of $x_i$, and consider the expression

$$\bar{E}(n)=\sum_{s=0}^{n}\bar{f}\left(\frac{s}{n}\right)F(n, s).$$

It can be shown, as in the last article, that

$$\lim_{n=\infty}\bar{E}(n)=0,$$

and by writing

$$\bar{E}(n)=\sum_{1}\bar{f}\left(\frac{s}{n}\right)F(n, s)+\cdots\cdots+\sum_{j-1}\bar{f}\left(\frac{s}{n}\right)F(n, s)$$
$$+\Sigma'\bar{f}\left(\frac{s}{n}\right)F(n, s),$$

where $\sum_1$, $\sum_2\cdots\cdots$,, $\sum_{j-1}$ have the same meaning as in the above and $\Sigma'$ is the summation about $s$ complementary to $\sum_1+\sum_2+\cdots\cdots+\sum_{j-1}$, we get

$$\lim_{n=\infty}\Sigma'\bar{f}\left(\frac{s}{n}\right)F(n, s)=0,$$

from which it also results that

$$\lim_{n=\infty}\Sigma' F(n, s)=0. \tag{21}$$

On the other hand if we consider the expression $E_i(n)$ defined by (13), then

$$\lim_{n=\infty}E_i(n)=\lambda_i f_i(x_i)\neq 0\text{(}^1\text{)}, \tag{22}$$

and by writing

$$E_i(n)=\sum_{1}f_i\left(\frac{s}{n}\right)F(n, s)+\cdots\cdots+\sum_{j-1}f_i\left(\frac{s}{n}\right)F(n, s)+\Sigma' f_i\left(\frac{s}{n}\right)F(n, s),$$

we have

$$\left|E_i(n)\right|\leqq\sum_{1}\left|f_i\left(\frac{s}{n}\right)\right|F(n, s)+\cdots\cdots+\sum_{j-1}\left|f_i\left(\frac{s}{n}\right)\right|F(n, s)$$
$$+\Sigma'\left|f_i\left(\frac{s}{n}\right)\right|F(n, s).$$

---

(1) The polynomial $f_i(x)$ may have imaginary coefficients.

Since, if $i \geqq j$ and $i_1 \leqq j-1$, $\left| f_i\left(\frac{s}{n}\right) \right|$ has a positive upper limit which vanishes with $\varepsilon$, for all $s$ and $n$ satisfying

$$\left| \frac{s}{n} - x_{i_1} \right| \leqq \varepsilon,$$

we shall obtain, just as in the last article,

$$\lim_{n=\infty} \sum_{i_1} \left| f_i\left(\frac{s}{n}\right) \right| F(n, s) = 0,$$

which, combined with (21), gives us

$$\lim_{n=\infty} E_i(n) = 0, \tag{23}$$

if $i \geqq j$. The relation (23) which has been obtained from the assumption that some of $x_1, x_2, \cdots\cdots, x_r$ satisfying (6) with the condition (20) do not lie in the interval $(0, 1)$ or are imaginary, is inconsistent with the relation (22)—a result showing the non-existence of the roots $x_j$, $x_{j+1}, \cdots\cdots$, and therefore

$$0 \leqq x_i \leqq 1, \qquad (i = 1, 2, \cdots\cdots, r) \tag{24}$$

thus establishing the validity of the condition (7).

**5.** It has been shown indirectly from the foregoing discussions that there can be no two different sets of values of $x_1, x_2, \cdots\cdots, x_r$; $\lambda_1, \lambda_2, \cdots\cdots, \lambda_r$ satisfying (6) with the condition (20)(1).

Again it is easily seen that in case when the system of the simultaneous equations (6) has solutions which are not subjected to the condition (20) either by the occurrence of equal $x_i$ or by the vanishing of some $\lambda_i$(2), we have always a system of the simultaneous equations

$$T_m = x_1^m \lambda_1 + x_2^m \lambda_2 + \cdots\cdots + x_{r'}^m \lambda_{r'}, \tag{25}$$
$$m = 0, 1, 2, \cdots\cdots, 2r', \qquad r' < r,$$

which has a solution subjecting to the condition (20).

For if the system (6), for a certain $r$, has a solution which does not satisfy the condition (20) there will exist also a solution of the system for $r$ diminished by 1 which does or does not satisfy the condition

(1) We consider as the same set of solutions those which can be obtained by the same permutations applied to $x_1, x_2, \cdots, x_r$ and $\lambda_1, \lambda_2, \cdots, \lambda_r$.

(2) In this case the system (6) will have infinitely many solutions.

(20), and for $r=1$ at least, if all the solutions in other cases do not satisfy (20), the system will have a unique solution $x_1=T_1$, $\lambda_1=1$.

Combining the results we are thus led to the conclusion:

*In order that a system of trials may be of the first class having r points of the first class it is necessary and sufficient that the simultaneous equations* (6) *should have a unique solution.*

*In case when the simultaneous equations have solutions more than one, they will have infinitely many solutions and the system of trials, in such a case, will be of the first class having $r'$ points $(r'<r)$ of the first class.*

The simplest case occurs when $r=1$. Since the system of the three equations

$$\left.\begin{aligned} 1&=\lambda_1 \\ T_1&=x_1\lambda_1 \\ T_2&=x_1^2\lambda_1 \end{aligned}\right\}$$

has a unique solution

$$x_1=T_1, \qquad \lambda_1=1,$$

if

$$T_2=T_1^2, \tag{26}$$

it follows that (26) *is the necessary and sufficient condition for the existence of a system of trials of the first class having one point of the first class.*

Again another simple case occurs when $r=2$ and

$$x_1=0, \qquad x_2=1.$$

Now in order that the system of the five equations

$$T_m=x_1^m\lambda_1+x_2^m\lambda_2,$$

$$m=0, 1, 2, 3, 4,$$

may have a unique solution of which

$$x_1=0 \qquad \text{and} \qquad x_2=1,$$

it is necessary and sufficient that

$$T_1=T_2=T_3=T_4, \qquad 0<T_1<1. \tag{27}$$

The condition (27) is equivalent to the mere condition

$$T_1=T_2, \qquad 0<T_1<1. \tag{28}$$

Indeed it can be shown that if

$$T_1 = T_2 = \cdots\cdots = T_{m-1}, \tag{29}$$

then

$$T_m = T_1.$$

For since

$$T_m\, T_{m-2} \geqq T_{m-1}^2,$$ [1]

while

$$T_m \leqq T_{m-1},$$

we shall have, if $T_m \neq T_{m-1}$, i.e. $T_m < T_{m-1}$,

$$T_{m-2} > T_{m-1},$$

which contradicts the assumption (29).

Consequently *the condition* (28) *is necessary and sufficient for the occurrence of a system of trials of the first class having two points* $x_1 = 0$ *and* $x_2 = 1$ *as the points of the first class;* and if this condition be satisfied we shall have

$$\lambda_1 = 1 - T_1, \qquad \lambda_2 = T_1.$$

**6.** The condition obtained in the last article may also be transformed as follows:—

If there exist a set of $x_1, x_2, \cdots\cdots, x_r$; $\lambda_1, \lambda_2, \cdots\cdots, \lambda_r$ satisfying (6) with the condition (20), then the set $A_1$ consists of $r$ points, all other points in the interval (0, 1) belonging to the set $A_0$ and hence by (5)

$$T_m = x_1^m\, \lambda_1 + x_2^m\, \lambda_2 + \cdots\cdots + x_r^m\, \lambda_r,$$

$$m = 1, 2, \cdots\cdots, 2r, 2r+1, \cdots\cdots$$

so that we have

$$\begin{aligned} \sum_{m=0}^{\infty} T_m x^m &= \lambda_1 \sum_{m=0}^{\infty} (x_1 x)^m + \lambda_2 \sum_{m=0}^{\infty} (x_2 x)^m + \cdots\cdots + \lambda_r \sum_{m=0}^{\infty} (x_r x)^m \\ &= \sum_{i=1}^{r} \frac{\lambda_i}{1 - x_i x} \qquad\qquad (30) \\ &= \frac{P(x)}{(1 - x_1 x)(1 - x_2 x) \cdots\cdots (1 - x_r x)}, \end{aligned}$$

where $P(x)$ is a polynomial of $x$ at most of the degree $r-1$, whence by writing

---

(1) This follows from the inequality

$$\sum_{s=0}^{n} a_s x_s^m \sum_{s=0}^{n} a_s x_s^{m-2} \geqq \left( \sum_{s=0}^{n} a_s x_s^{m-1} \right)^2$$

where $a$ and $x_s$ are all positive.

$$(x-x_1)(x-x_2)\cdots\cdots(x-x_r)$$
$$=x^r-b_1x^{r-1}+b_2x^{r-2}-\cdots\cdots+(-1)^r b_r,$$

we get

$$(1-b_1x+b_2x^2-\cdots\cdots+(-1)^r b_r x^r).\sum_{m=0}^{\infty} T_m x^m=P(x).$$

Hence equating the coefficients of $x^r, x^{r+1},\cdots\cdots, x^{2r}$ in both sides,

$$\left.\begin{array}{l} T_r-b_1T_{r-1}+b_2T_{r-2}-\cdots\cdots+(-1)^r b_r=0,\\ T_{r+1}-b_1T_r+b_2T_{r-1}-\cdots\cdots+(-1)^r b_r T_1=0,\\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\ T_{2r}-b_1T_{2r-1}+b_2T_{2r-2}-\cdots+(-1)^r b_r T_r=0. \end{array}\right\} \quad (31)$$

Conversely we may derive the last $r+1$ equations of (6) from its first $r$ equations together with the equations (31), so that the system (6) is equivalent to the system of equations consisting of the first $r$ equations of (6) and the equations (31).

Now the equations (31) only involve $x_1, x_2,\cdots\cdots, x_r$ but not $\lambda_1, \lambda_2, \cdots\cdots, \lambda_r$. Hence in order that the system (6) may have a unique solution it is necessary that the equations (31) should have also a unique solution with respect to $b_1, b_2, \cdots\cdots, b_r$, for which the necessary and sufficient condition is that the two matrices

$$M_1=\begin{pmatrix} T_r & T_{r-1} & \cdots\cdots & 1\\ T_{r+1} & T_r & \cdots\cdots & T_1\\ \cdots & \cdots & \cdots & \cdots\\ T_{2r} & T_{2r-1} & \cdots\cdots & T_r \end{pmatrix} \quad \text{and} \quad M_2=\begin{pmatrix} T_{r-1} & T_{r-2} & \cdots\cdots & 1\\ T_r & T_{r-1} & \cdots\cdots & T_1\\ \cdots & \cdots & \cdots & \cdots\\ T_{2r-1} & T_{2r-2} & \cdots\cdots & T_r \end{pmatrix}$$

have the same rank $r$. From this we may infer that

$$\Delta_{r+1}=\begin{vmatrix} T_r & T_{r-1} & \cdots\cdots & 1\\ T_{r+1} & T_r & \cdots\cdots & T_1\\ \cdots & \cdots & \cdots & \cdots\\ T_{2r} & T_{2r-1} & \cdots\cdots & T_r \end{vmatrix}=0, \quad \Delta_r=\begin{vmatrix} T_{r-1} & T_{r-2} & \cdots\cdots & 1\\ T_r & T_{r-1} & \cdots\cdots & T_1\\ \cdots & \cdots & \cdots & \cdots\\ T_{2r-2} & T_{2r-3} & \cdots\cdots & T_{r-1} \end{vmatrix}\neq 0(^1). \quad (32)$$

(1) That the first determinant $\Delta_{r+1}$ vanishes is evident. As to the second determinant $\Delta_r$, if it also vanishes, the rank of the matrix

$$M_3=\begin{pmatrix} T_r & T_{r-1} & \cdots & 1\\ \cdots & \cdots & \cdots & \cdots\\ T_{2r-1} & T_{2r-2} & \cdots & T_{r-1} \end{pmatrix}$$

Moreover, in order that the system (6) may have a unique solution satisfying (20) the equation

$$\Delta_r(x)=\begin{vmatrix} x^r & x^{r-1} \cdots\cdots 1 \\ T_r & T_{r-1} \cdots\cdots 1 \\ \cdots\cdots\cdots\cdots & \cdots\cdots \\ T_{2r-1} & T_{2r-2} \cdots\cdots T_{r-1} \end{vmatrix}=0, \tag{33}$$

which is obtained by the elimination of $b_1, b_2, \cdots\cdots, b_r$ from the first $r$ equations of (31) and

$$x^r-b_1x^{r-1}+\cdots\cdots+(-1)^r b_r=0,$$

must have $r$ different roots.

Thus we have obtained as necessary conditions for the existence of a unique solution of (6) the relation (32) and also the non-existence of multiple roots of the equation (33).

That the conditions are sufficient for the existence of a unique solution of (6) can at once be seen.

Consequently we have the result:

*In order that a system of trials may be of the first class having $r$ points of the first class it is necessary and sufficient that*

$$\Delta_{r+1}=0, \qquad \Delta_r \neq 0,$$

*and the equation* (33) *should have no multiple roots*( [1] ).

*If this condition be satisfied the points of the first class and their corresponding probabilities $\lambda_i$ may be found by solving the equations* (33) *and then the first $r$ equations of* (6)( [2] ).

---

must be less than $r$ for otherwise the system of the first $r$ equations of (31) will be inconsistent. Since the two matrices $M_2$ and $M_3$ have clearly the same rank this contradicts the fact that the rank of $M_2$ is $r$, a result showing the non-vanishing of the second determinant $\Delta_r$.

It is also easy to see that

$$\Delta_{r+2}=\Delta_{r+3}=\cdots\cdots=0$$

in this case.

( [1] ) It can be shown that the last condition is superfluous, i.e. it is a direct consequence from the non-vanishing of $\Delta_r$. See the Appendix § 1.

( [2] ) $\lambda_i$ may be found more simply by aid of (15). Indeed if $f_i(x)$ be expressed in the form

$$f_i(x)=x^{r-1}+a_1' x^{r-2}+\cdots\cdots+a'_{r-1},$$

the relation (15) will give us

$$T_{r-1}+a_1' T_{r-2}+\cdots\cdots+a'_{r-1}=\lambda_i f_i(x_i)$$

From the discussions here given it is also easy to derive another criterion which runs as follows:

*In order that a system of trials may be of the first class having $r$ points of the first class it is necessary and sufficient that the function*

$$\phi_1^1(z) = \sum_{m=0}^{\infty} T_m z^m \tag{34}$$

*is a rational function of $z$ either having poles of the first order at $r$ different points and vanishing with $\frac{1}{z}$*(1)*, or having poles of the first order at $r-1$ points and taking a determinate finite value different from zero at $z=\infty$* (2).

**7.** We have investigated, in the last articles, conditions to be satisfied by $T_1, T_2, T_3, \cdots\cdots$ for the occurrence of a system of trials of the first class. To apply the conditions to practical problems we need to calculate $T_1, T_2, \cdots\cdots$ beforehand.

It has been shown, in my previous paper(3), that if $p_m$ denote the probability of the happening of the event under consideration in $m$th trial *à priori* and

$$W_{m_1, m_2, \cdots\cdots}, \qquad m_1 < m_2 < \cdots\cdots$$

the probability of the concurrence of the event in $m_1$th, $m_2$th,$\cdots\cdots$trials, the occurrence or non-occurrence of the event in other trials being perfectly disregarded, then

$$\sum_{s=0}^{n} F(n, s)\, y^s = 1 + P_1(n)(y-1) + P_2(n)(y-1)^2 + \cdots\cdots + P_n(n)(y-1)^n, \tag{35}$$

where

$$\left.\begin{aligned} P_1(n) &= p_1 + p_2 + \cdots\cdots + p_n, \\ P_2(n) &= W_{1,2} + W_{1,3} + \cdots\cdots + W_{n-1,n}, \\ P_3(n) &= W_{1,2,3} + \cdots\cdots\cdots + W_{n-2,n-1,n}, \\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ P_n(n) &= W_{1,2,3,\cdots,n}. \end{aligned}\right\} \tag{36}$$

This formula now serves us for the calculation of $T_1, T_2, \cdots\cdots$.

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(1) This case arises when none of $x_i$ $(i=1, 2, \cdots, r)$ vanishes.

(2) This case arises when one of $x_i$ vanishes.

(3) L. C., p. 28. The notation $W_{m_1, m_2, \cdots}$ here used is somewhat different from that there employed.

If we differentiate both sides of (35) with respect to $y$ $m$ times and then put $y=1$, we obtain

$$\sum_{s=0}^{n} s(s-1)\cdots\cdots(s-m+1)\,F(n,s)=m!\,P_m(n),$$

which gives

$$T_m=\lim_{n=\infty}\sum_{s=0}^{n}\left(\frac{s}{n}\right)^m F(n,s)=m!\lim_{n=\infty}\frac{P_m(n)}{n^m}, \tag{37}$$

*a formula by means of which $T_m$ can be calculated.*

The conditions obtained in the last articles may therefore be transformed into one which $P_m(n)\,(m=1,2,\cdots\cdots)$ will satisfy.

For example the condition (26) is expressed in the form

$$\left.\begin{array}{ll}\text{(i)} & \lim\limits_{n=\infty}\dfrac{p_1+p_2+\cdots\cdots+p_n}{n}=p \quad \text{exists} \\[2ex] \text{(ii)} & \lim\limits_{n=\infty}\dfrac{2P_2(n)-(p_1+p_2+\cdots\cdots+p_n)^2}{n^2}=0,\end{array}\right\} \tag{38}$$

of which the latter may be replaced by

$$\lim_{n=\infty}\frac{V}{n^2}=0, \tag{39}$$

where

$$V=\sum(W_{k,l}-p_k p)\qquad \begin{array}{l}k=1,2,\cdots\cdots,l-1,\\ l=2,3,\cdots\cdots,n.\end{array}$$

Similarly the condition (28) is expressed in the form

$$\left.\begin{array}{ll}\text{(i)} & \lim\limits_{n=\infty}\dfrac{p_1+p_2+\cdots\cdots+p_n}{n}=p \quad \text{exists} \\[2ex] \text{(ii)} & \lim\limits_{n=\infty}\dfrac{2P_2(n)-(p_1+p_2+\cdots\cdots+p_n)^2}{n^2}=p(1-p)\neq 0,\end{array}\right\} \tag{40}$$

of which the latter may also be replaced by

$$\lim\frac{V}{n^2}=\frac{1}{2}pq\neq 0, \tag{41}$$

where $V$ stands for the same expression as in the above and

$$q=1-p.$$

**8.** Some examples of the system of non-independent trials satisfy-

ing the condition (39) have been given in my previous paper. I shall take this occasion to solve a more general example than that there given.

We thus consider *the case where the two persons A and B play a set of games each of which can not be drawn and in which A's chance of winning in the first game is* $p_1$ *but in the mth game it is* $P_m$ *if he wins the game before and is* $Q_m$ *if he loses it,* $P_m$ *and* $Q_m$ *being subjected to the conditions*

$$\lim_{m=\infty} m^a (1-P_m)=P', \qquad \lim_{m=\infty} m^a Q_m=Q, \tag{42}$$

*where a is zero or a positive number, and* $P'$, $Q$ *are two finite numbers one at least of which does not vanish.*

In this example $A$'s chance $p_m$ of winning in the $m$th game is given by

$$\begin{aligned} p_m &= p_{m-1} P_m + (1-p_{m-1}) Q_m \text{(}^1\text{)} \\ &= p_{m-1} (P_m - Q_m) + Q_m. \end{aligned} \tag{43}$$

Similarly $A$'s chance $W_{m_1, m}$ of winning both in the $m_1$th and $m$th games is given by

$$W_{m_1, m} = W_{m_1, m-1} P_m + (p_{m_1} - W_{m_1, m-1}) Q_m, \quad (m_1 \leqq m-2), \tag{44}$$

and hence, by aid of (43),

$$W_{m_1, m} - p_{m_1} p_m = (P_m - Q_m) \{ W_{m_1, m-1} - p_{m_1} p_{m-1} \},$$

from which by putting

$$V_{m_1, m} = W_{m_1, m} - p_{m_1} p_m, \tag{45}$$

we obtain

$$V_{m_1, m} = (P_m - Q_m) V_{m_1, m-1}. \tag{46}$$

When $m_1 = m-1$ we have, in place of (46),

$$V_{m-1, m} = p_{m-1} (P_m - p_m).$$

Consequently again by putting

$$V_m = \sum_{m_1=1}^{m-1} V_{m_1, m},$$

we get

$$V_m = (P_m - Q_m) V_{m-1} + p_{m-1} (P_m - p_m). \tag{47}$$

---

(1) Of course $P_m$, $Q_m$ and $p_m$ all lie between 0 and 1 both limits inclusive.

Now we shall first consider the case when

$$a=0.$$

If the two limits

$$\lim (1-P_m) \quad \text{and} \quad \lim Q_m$$

exist, then, since $0 \leqq p_m \leqq 1$, i.e. the sequence $(p_m)$ is bounded, we get by (43)

$$\lim_{m=\infty} \{ p_m-(P-Q)\, p_{m-1} \} = Q,$$

where

$$P=\lim_{m=\infty} P_m,$$

and therefore by Stolz's theorem on the limit of quotient we have

$$\lim_{n=\infty} \frac{\sum\limits_2^n p_m-(P-Q) \sum\limits_2^n p_{m-1}}{n} = Q,$$

so that, in virtue of the supposition $P'+Q>0$ namely $P-Q \neq 1$,

$$\lim_{n=\infty} \frac{\sum\limits_2^n p_m}{n} = \frac{Q}{1-P+Q} \text{ (1)}. \tag{48}$$

Similarly, since $\left| \frac{V_{m-1}}{m} \right| < 1$, we obtain from the expression (47) for $V_m$

$$\lim_{m=\infty} \frac{V_m-(P-Q)\, V_{m-1}}{m} = 0,$$

and therefore we shall have

$$\lim_{n=\infty} \frac{\sum\limits_3^n V_m-(P-Q) \sum\limits_3^n V_{m-1}}{n^2} = 0,$$

which gives as in the above

$$\lim_{n=\infty} \frac{\sum\limits_3^n V_m}{n^2} = 0,$$

and accordingly, by putting $V=\sum\limits_3^n V_m$,

(1) In case when $|P-Q|<1$, this is also an immediate consequence from Kubota's modification of Pólya's theorem [Tôhoku Math. Journ., Vol. 12 (1917) p. 222.].

$$\lim_{n=\infty} \frac{V}{n^2} = 0. \tag{49}$$

The two relations (48) and (49) show us that the condition (38) is satisfied for our case and therefore we have the result:

*The set of games is of the first class, with respect to A's winning, having a single point* $\frac{Q}{P'+Q}$ *of the first class in case when* $\alpha=0$.

As a special case, when $Q=0$ and therefore $P' \neq 0$ we have the point $x=0$ of the first class; and when $P'=0$ and therefore $Q \neq 0$ we get the point $x=1$ of the first class.

We shall next consider the case when

$$\alpha > 0.$$

In order to calculate $T_1$, $T_2$ in this case we establish the following two lemmas relating to the sequence.

*Lemma I. If the sequence* $(v_n)$ *satisfies*

$$n^\alpha (v_n - v_{n-1}) + c\, v_{n-1} \to l, \tag{50}$$

*where* $c$ *and* $l$ *are real numbers of which the former is positive and*

$$\alpha > 0,$$

*then the sequence is convergent.*

*Lemma II. The sequence* $(v_n)$ *thus defined has the limit* $\frac{l}{c}$ *in case when*

$$0 < \alpha \leqq 1 \text{ (1)}.$$

To prove the lemma I it will be sufficient to consider the case when the sequence $(v_m, v_{m+1}, \cdots\cdots)$ is not monotonic, however large $m$ may be taken(2).

Now if we put

$$n^\alpha (v_n - v_{n-1}) + c\, v_{n-1} = l_n,$$

we have

(1) The case when $\alpha=1$ is a direct consequence from Mercer's theorem. Mercer, Proc. of the London Math. Soc., series 2, Vol. 5, (1907) p. 207. Narumi, Tôhoku Math. Journ. Vol. 12 (1917) p. 288.

(2) For the sequence is bounded if $l_n \to l$, as is easily seen from the relations (52).

$$v_n - v_{n-1} = \frac{c}{n^a}\left(\frac{l_n}{c} - v_{n-1}\right), \tag{51}$$

so that, if $n$ be taken larger than $c^{\frac{1}{a}}$, we get two alternatives

$$\left.\begin{aligned} & v_{n-1} \leqq v_n \leqq \frac{l_n}{c}, \\ \text{or} \quad & v_{n-1} \geqq v_n \geqq \frac{l_n}{c}. \end{aligned}\right\} \tag{52}$$

If the sequence $(v_m, v_{m+1}, \cdots\cdots)$ is not monotonic, however large $m$ may be taken, then there will exist a sequence of integers $(n_1, n_2, \cdots\cdots)$ such that

$$\begin{aligned} & v_{n_1} \leqq v_{n_1+1} \leqq \cdots\cdots \leqq v_{n_2}, \\ & v_{n_2} \geqq v_{n_2+1} \geqq \cdots\cdots \geqq v_{n_3}, \\ & v_{n_3} \leqq v_{n_3+1} \leqq \cdots\cdots \leqq v_{n_4}, \\ & \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \end{aligned}$$

If this be the case we have, in virtue of (52),

$$\begin{aligned} & v_{n_2} \leqq \frac{l_{n_2}}{c}, \qquad v_{n_4} \leqq \frac{l_{n_4}}{c}, \quad \cdots\cdots \\ & v_{n_3} \geqq \frac{l_{n_3}}{c}, \qquad v_{n_5} \geqq \frac{l_{n_5}}{c}, \quad \cdots\cdots \end{aligned}$$

and therefore

$$\overline{\lim_{n=\infty}}\, v_n \leqq \frac{l}{c} \quad \text{and} \quad \underline{\lim_{n=\infty}}\, v_n \geqq \frac{l}{c},$$

which can only be satisfied when

$$v_n \to \frac{l}{c},$$

thus establishing the lemma I.

Again to prove the lemma II we put

$$\theta_n = \frac{l_n}{c} - v_{n-1},$$

so that $\lim\limits_{n=\infty} \theta_n = k$ exists. Now if $k \neq 0$ then by taking a positive number $\varepsilon$ such that $k-\varepsilon$ and $k+\varepsilon$ have the same sign as $k$ we may choose $m$ so large as to ensure that

$$k-\varepsilon<\theta_n<k+\varepsilon, \quad \text{for} \quad n\geqq m.$$

The relation (51) then gives

$$c\,(k-\varepsilon)\sum_{n=m}^{m+l}\frac{1}{n^a}<v_{m+l}-v_{m-1}<c(k+\varepsilon)\sum_{n=m}^{m+l}\frac{1}{n^a},$$

for any integer $l$. Since the series $\sum_{n=m}^{\infty}\frac{1}{n^a}$ is divergent if $0<a\leqq 1$, and $c\,(k-\varepsilon)$ and $c\,(k+\varepsilon)$ have the same sign, this contradicts the fact that $\lim_{l=\infty} v_{m+l}$ exists. The assumption $k\neq 0$ is inadmissible in the case when $0<a\leqq 1$ and hence

$$v_n\to\frac{l}{c}\,(^1),$$

so as to show the truth of the lemma II.

We shall now apply the lemmas to the calculation of $T_1$ and $T_2$.

Since we have by (43)

$$m^a(p_m-p_{m-1})+m^a(1-P_m+Q_m)\,p_{m-1}=m^a\,Q_m,$$

and the sequence $(p_{m-1})$ is bounded it follows that

$$m^a\,(p_m-p_{m-1})+(P'+Q)\,p_{m-1}\to Q,$$

and therefore by the lemma I

$$T_1=\lim_{m=\infty}p_m=p \tag{53}$$

exists in our case. Specially when $0<a\leqq 1$, we have

$$T_1=\frac{Q}{P'+Q}. \tag{54}$$

Again by putting $\qquad \frac{V_m}{m}=v_m,$

so that the sequence $(v_m)$ is bounded we get from (47)

$$m(v_m-v_{m-1})+\{1+(m-1)(P_m'+Q_m)\}v_{m-1}=p_{m-1}(P_m-p_m), \tag{55}$$

---

(1) It may happen that $v_n$ does not tend to $\frac{l}{c}$ in case when $a>1$ if the sequence $(v_n)$ be monotonic, but if the sequence $(v_n)$ be not monotonic, however large $n$ may be taken, then we have also

$$v_n\to\frac{l}{c}$$

as is easily seen from the foregoing discussion.

where

$$P'_m = 1 - P_m,$$

and therefore if $\alpha > 1$, i.e. $\lim_{m=\infty} m(P'_m + Q_m) = 0$, then

$$m(v_m - v_{m-1}) + v_{m-1} \to p(1-p);$$

and if $\alpha = 1$, i.e. $\lim_{m=\infty} m(P'_m + Q_m) = P' + Q$, then

$$m(v_m - v_{m-1}) + (1 + P' + Q) v_{m-1} \to p(1-p). \qquad (56)$$

Also if $0 < \alpha < 1$, dividing the both sides of the relation (55) by $m^{1-\alpha}$ and then making $m$ infinite,

$$m^\alpha (v_m - v_{m-1}) + (P' + Q) v_{m-1} \to 0.$$

Consequently by the lemma II, putting $q = 1 - p$,

$$\begin{aligned} v_m &\to pq && \text{if} \quad \alpha > 1 \\ &\to \frac{pq}{1 + P' + Q} && \text{if} \quad \alpha = 1 \\ &\to 0 && \text{if} \quad 0 < \alpha < 1. \end{aligned}$$

Since, by Stolz's theorem,

$$\lim_{n=\infty} \frac{V}{n^2} = \lim_{m=\infty} \frac{v_m}{2},$$

provided that the right-hand limit exists, we have

$$\lim_{n=\infty} \frac{V}{n^2} = \frac{1}{2} pq \qquad \text{if} \quad \alpha > 1 \qquad (57)$$

$$= \frac{pq}{2(1 + P' + Q)} \qquad \text{if} \quad \alpha = 1 \qquad (58)$$

$$= 0 \qquad \text{if} \quad 0 < \alpha < 1. \qquad (59)$$

The two relations (53) and (59) show us that the condition (38) is satisfied if $0 < \alpha < 1$ and hence, combining with the result obtained for $\alpha = 0$, we arrive at:

*The set of games is of the first class having a single point* $\dfrac{Q}{P' + Q}$ *of the first class in case when* $0 \leqq \alpha < 1$.

Also the two relations (53) and (57) show us that, if $\alpha > 1$, the condition (40) is to be satisfied unless $p = 0$ or $p = 1$.

The value of $p$ depends on $p_1$, $P_m$ and $Q_m$ and it may be zero or

1 or any other intermediate number between 0 and 1. For example

if $$p_1=0, \quad P_m=1, \quad Q_m=\frac{2}{(m+1)(m+2)},$$

then $$p_m=\frac{1}{2}-\frac{1}{m+1} \quad \text{and therefore} \quad p=\frac{1}{2},$$

while if $$p_1=0, \quad P_m=1, \quad Q_m=\frac{3}{(m+2)(2m+5)},$$

then $$p_m=\frac{1}{3}-\frac{1}{m+2} \quad \text{and therefore} \quad p=\frac{1}{3},$$

and so on. Consequently we have the result:

*The set of games is of the first class having two points* 0 *and* 1 *of the first class, in case when* $\alpha>1$, *if the limit of* $p_m$ *which necessarily exists be different from* 0 *and* 1 (1).

In case when $\alpha=1$, the two relations (54) and (58) show us that if both of $P'$ and $Q$ do not vanish then $T_1$ is different from zero and 1 and neither of the conditions (38) nor (40) can be satisfied. It requires therefore further investigation to determine whether the set of games belongs to the class considered. It will be shown in the Part II that this case belongs to a class of another different character and hence we shall not enter here into its detailed discussions.

**9.** Before leaving the present Part we shall shortly consider the function $\overline{\lambda}(x)$ defined by

$$\overline{\lambda}(x)=\pi(0, x). \tag{60}$$

It is evident that every point of the set $A_0$ is a point of linear invariability(2) of the function $\overline{\lambda}(x)$, and at points of the set $A_1$ the function $\overline{\lambda}(x)$ may become indeterminate or at least it becomes discontinuous, the saltus of the function at a point $x=x_\mu$ of the set $A_1$ being equal to $\pi(x_\mu; \varepsilon)$, i.e. $\lambda_\mu$. Consequently if all the points $x_1, x_2, \cdots\cdots, x_r$ of the set $A_1$ be arranged in order of increasing magnitude then

(1) It can be shown that if both of $P'$ and $Q$ do not vanish then $0<p<1$. But even if any one of $P'$ and $Q$ vanishes still it may happen that $0<p<1$ as is evident from the above examples.

(2) If a neighbourhood $(x-\delta, x+\delta)$ of the point $x$ can be found by taking $\delta$ sufficiently small, which is such that $\lambda(x)$ has the same value at all points in the neighbourhood, then the point $x$ is called a *point of linear invariability* of the function.

$$\left.\begin{aligned}\overline{\lambda}(x)&=0 && \text{if} \quad 0\leqq x<x_1\\ &=\sum_{i=1}^{\mu}\lambda_i && \text{if} \quad x_\mu<x<x_{\mu+1}\\ &=1 && \text{if} \quad x_r<x\leqq 1.\end{aligned}\right\} \tag{61}$$

Now we shall show that *the function* $\overline{\lambda}(x)$ *may be expressed, at points of the set* $A_0$ *not coinciding with* 0 *or* 1, *in the form*

$$\overline{\lambda}(x)=\frac{1}{\pi}\int_0^\infty\left[\left\{\sum_{m=0}^{\infty}(-1)^m\frac{T'_{2m+1}}{(2m+1)!}y^{2m}\right\}\cos xy\right.$$
$$\left.+\left\{\sum_{m=0}^{\infty}(-1)^{m+1}\frac{T'_{2m}}{(2m)!}y^{2m-1}\right\}\sin xy\right]dy\text{(}^1\text{)}, \tag{62}$$

where

$$T'_m=1-T_m.$$

Since $T'_m$ satisfies, for our present case,

$$T'_m=1-(x_1^m\lambda_1+x_2^m\lambda_2+\cdots\cdots+x_r^m\lambda_r),\quad m=0,1,2,\cdots\cdots,$$

we obtain

$$\sum_{m=0}^{\infty}(-1)^m\frac{T'_{2m+1}}{(2m+1)!}y^{2m}=\frac{\sin y-\sum_{i=1}^{r}\lambda_i\sin x_iy}{y},$$

$$\sum_{m=0}^{\infty}(-1)^{m+1}\frac{T'_{2m}}{(2m)!}y^{2m-1}=\frac{-\cos y+\sum_{i=1}^{r}\lambda_i\cos x_iy}{y},$$

so that denoting the right-hand expression of (62) by $\Delta$ we have

$$\Delta=\frac{1}{\pi}\int_0^\infty\left\{\frac{\sin(1-x)y}{y}-\sum_{i=1}^{r}\frac{\lambda_i\sin(x_i-x)y}{y}\right\}dy,$$

and therefore, if $x_\mu<x<x_{\mu+1}$,

$$\Delta=\frac{1}{\pi}\left\{\frac{\pi}{2}+\frac{\pi}{2}\left(\sum_{i=1}^{\mu}\lambda_i-\sum_{i=\mu+1}^{r}\lambda_i\right)\right\}$$

which, combined with

$$1=\sum_{i=1}^{r}\lambda_i,$$

gives

(1) It will later be shown that this relation holds good for a more general case.

$$\Delta = \sum_{i=1}^{p} \lambda_i \text{ (1)},$$

a result showing the validity of the relation (62).

## Part II.

## System of Trials of the generalized zero Class with respect to an Event.

**10.** We now proceed to consider a system of trials of a quite different character.

In the Introduction, $\pi(x-\varepsilon, x+\varepsilon')$ has been defined as the limit, when $n=\infty$, of the probability $\pi_n(x-\varepsilon, x+\varepsilon')$ that $s$ will satisfy

$$x-\varepsilon \leqq \frac{s}{n} \leqq x+\varepsilon',$$

where $s$ denotes the number of the happening of an event in $n$ trials. Now a point $x$ in the interval $(0, 1)$ may be such that the limiting probability $\pi(x-\varepsilon, x+\varepsilon')$ is a continuous function of $\varepsilon$ and $\varepsilon'$ within

$$0<\varepsilon \leqq \delta, \quad 0<\varepsilon' \leqq \delta,$$

by taking a positive number $\delta$ sufficiently small and also

$$\lim_{\varepsilon, \varepsilon'=+0} \pi(x-\varepsilon, x+\varepsilon')=0.$$

Let us call such a point $x$ the point of the *generalized zero class* and let the set of such points be denoted by $A_0'$. It is evident that the set $A_0$ defined in §1 is a component of the set $A_0'$.

*If $x$ be a point of the set $A_0'$, then*

$$\pi(x, x)=0 \text{ (2)}. \tag{63}$$

For since

(1) Of course this is to be replaced by zero if $0<x<x_1$ and by 1 if $x_p<x<1$ provided that 0 and 1 belong to the set $A_0$. Also in this case $\Delta$ has the value 0 when $x=0$ and $\frac{1}{2}$ when $x=1$.

(2) This is defined as the limit, when $n=\infty$, of the probability that $s$ will satisfy

$$\frac{s}{n}=x.$$

$$\pi_n(x-\varepsilon, x+\varepsilon') \geqq \pi_n(x, x),$$

we have

$$\overline{\lim_{n=\infty}}\, \pi_n(x, x) \leqq \pi(x-\varepsilon, x+\varepsilon'),$$

of which the right-handside may be made as small as we please by taking two positive numbers $\varepsilon$ and $\varepsilon'$ sufficiently small. It follows that $\pi(x, x)$ exists and vanishes.

Now let us show that

*If all points within the closed interval* $(\alpha \leqq x \leqq \beta)$ *belong to the set* $A_0'$ *then the function* $\pi(\alpha, x)$ *is a continuous function of* $x$ *within the same closed interval or rather within a wider interval.*

For suppose that this is not the case. Then since $\pi(\alpha, x)$ is a continuous function of $x$ at least in a sufficiently small neighbourhood of $x=\alpha$, if we consider the set of points $(x_1)$ up to which the limiting probability $\pi(\alpha, x)$ is continuous, the number $x_1$ must have an upper limit $k$ not greater than $\beta$.

We have, for the point $x=k$,

$$\lim_{\varepsilon, \varepsilon'=+0} \pi(k-\varepsilon, k+\varepsilon')=0,$$

while, in virtue of $\pi(k-\varepsilon, k-\varepsilon)=0$,

$$\pi(\alpha, k+\varepsilon')=\pi(\alpha, k-\varepsilon)+\pi(k-\varepsilon, k+\varepsilon'). \tag{64}$$

Consequently we get

$$\lim_{\varepsilon'=+0} \pi(\alpha, k+\varepsilon')=\lim_{\varepsilon=+0} \pi(\alpha, k-\varepsilon).$$

Since evidently

$$\pi(\alpha, k+\varepsilon') \geqq \pi(\alpha, k) \geqq \pi(\alpha, k-\varepsilon),$$

it follows that the function $\pi(\alpha, x)$ is continuous at the point $x=k$.

Moreover we see from (64) that $\pi(\alpha, k+\varepsilon')$ is a continuous function of $\varepsilon'$ in a sufficiently small positive neighbourhood of $\varepsilon'=0$(1) namely within $0<\varepsilon' \leqq \delta$. It then follows that the function $\pi(\alpha, x)$ is continuous within the interval $(\alpha, k+\delta)$ which contradicts the assumption that $k$ is the upper limit of the set $(x_1)$. The supposition is therefore inadmissible and hence we have the theorem.

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(1) "Positive neighbourhood" is used in the sense of "neighbourhood on the right."

From this theorem it results also at once that

*If all points within the closed interval* $(\alpha' \leqq x \leqq \beta)$, *where* $\alpha < \alpha'$, *belong to the set* $A_0'$ *then the function* $\pi(\alpha, x)$ *is a continuous function of* $x$ *within an interval containing the same closed interval, provided that* $\pi(\alpha, \alpha')$ *exists.*

*If all points within the open interval* $(\alpha' < x < \beta)$ *where* $\alpha < \alpha'$ *belong to the set* $A_0'$ *then the functions* $\pi(\alpha, x)$ *is a continuous function of* $x$ *within the same open interval, provided that* $\pi(\alpha, x)$ *exists at least at a point within it.*

Let us next prove the converse theorem that

*If* $\pi(\alpha, x)$ *is a continuous function of* $x$ *within the closed or open interval* $(\alpha, \beta)$ *then all points within the open interval* $(\alpha < x < \beta)$(1) *belong to the set* $A_0'$.

If $X$ be a point in the open interval $(\alpha, \beta)$ it is easy to see that

$$\pi(X, X) = 0, \tag{65}$$

from the continuity of the function $\pi(\alpha, x)$ at the point $x = X$. Hence if $x_1$ be a point in the same open interval then by taking two positive numbers $\varepsilon_1$ and $\varepsilon_1'$ such that

$$\alpha < x_1 - \varepsilon_1 < x_1 + \varepsilon_1' < \beta, \tag{66}$$

we have

$$\pi(x_1 - \varepsilon_1, x_1 + \varepsilon_1') = \pi(\alpha, x_1 + \varepsilon_1') - \pi(\alpha, x_1 - \varepsilon_1). \tag{67}$$

Since the function $\pi(\alpha, x)$ is a continuous function of $x$ in a sufficiently small neighbourhood of $x = x_1$, the right-hand side expression and therefore also the left-hand side one of (67) is a continuous function of $\varepsilon_1$ and $\varepsilon_1'$ in a sufficiently small positive neighourhood of $\varepsilon_1 = 0$ and $\varepsilon_1' = 0$, which vanishes at both of them. The point $x_1$ therefore belongs to the set $A_0'$.

From this theorem it follows also that

*If* $x$ *be a point of the set* $A_0'$ *such that* $\pi(x - \varepsilon, x + \varepsilon')$ *is a continuous function of* $\varepsilon$ *and* $\varepsilon'$ *within* $0 < \varepsilon \leqq \delta$, $0 < \varepsilon' \leqq \delta$ *then all points within the open interval* $(x - \delta, x + \delta)$ *belong to the set* $A_0'$.

For $\pi(x - \varepsilon, x + \varepsilon')$ may be expressed in the form

$$\pi(x - \varepsilon, x + \varepsilon') = \pi(x - \varepsilon, x) + \pi(x, x + \varepsilon'),$$

so that $\pi(x - \varepsilon, x)$ and $\pi(x, x + \varepsilon')$ are respectively continuous functions

(1) But not necessarily within the closed interval $(\alpha \leqq x \leqq \beta)$.

of $\varepsilon$ and $\varepsilon'$ within $0<\varepsilon\leqq\delta$ and $0<\varepsilon'\leqq\delta$ if $\pi(x-\varepsilon, x+\varepsilon')$ be so.

This result shows us that in a sufficiently small neighbourhood of a point of the set $A_0'$ there exists no point which does not belong to the set $A_0'$ and therefore

*The points complementary to $A_0'$ in the closed interval (0,1) form a closed set, if such points exist.*

But we can not infer that there always exists at least a point complementary to $A_0'$. Indeed there may exist a system of trials for which all the points in the interval (0, 1) belong to the set $A_0'$. Such a system of trials is said to be of the *generalized zero class* with respect to the event under consideration.

Since for such a system of trials, $\pi(0,x)$ is a continuous non-decreasing function of $x$ in the interval (0, 1) it may happen that it is expressible in the form

$$\pi(0, x)=\int_0^x \lambda(x)\,dx\,, \tag{68}$$

where $\lambda(x)$ is a non-negative function of $x$ which possesses at least the convergent improper integral in the interval (0, 1). In the first place we shall derive conditions for the occurrence of this special case, leaving for later discussions the general case where $\pi(0,x)$ is not expressible in such a form.

**11.** We first begin with the necessary conditions.

If the function

$$\overline{\lambda}(x)=\pi(0, x)$$

be expressible in the form

$$\overline{\lambda}(x)=\int_0^x \lambda(x)\,dx\,,$$

in which $\lambda(x)$ is a non-negative integrable function in the interval (0, 1) then all the functions $x^m\lambda(x)$ $(m>0)$ are also integrable in the same interval. Hence in such a case by dividing the whole interval (0, 1) into $k$ equal sub-intervals $\delta_1, \delta_2, \cdots\cdots, \delta_k$ of which $\delta_r$ has the end points $x_{r-1}$ and $x_r$ (1) where $x_0=0$, $x_k=1$, we have

$$\sum_{r=1}^{k} x_{r-1}^m \int_{x_{r-1}}^{x_r} \lambda(x)\,dx \leqq \int_0^1 x^m \lambda(x)\,dx \leqq \sum_{r=1}^{k} x_r^m \int_{x_{r-1}}^{x_r} \lambda(x)\,dx\,,$$

(1) Here and hereafter we shall always assume that $x_{r-1}<x_r$ in case when they are taken for the end points of an interval.

i. e.

$$\sum_{r=1}^{k} x_{r-1}^{m}\,\pi\,(x_{r-1}, x_r) \leqq \int_0^1 x^m\,\lambda\,(x)\,dx \leqq \sum_{r=1}^{k} x_r^m\,\pi\,(x_{r-1}, x_r)\,. \quad (69)$$

On the other hand from the relation

$$\sum_{r=1}^{k} x_{r-1}^{m}\,\pi_n\,(x_{r-1}, x_r) \leqq \sum_{s=0}^{n}\left(\frac{s}{n}\right)^m F(n, s) + \sum_{r=1}^{k-1} x_r^m\,\pi_n\,(x_r, x_r)$$

$$\leqq \sum_{r=1}^{k} x_r^m\,\pi_n\,(x_{r-1}, x_r)\,,$$

where $F(n, s)$ denotes the probability of the happening of the event under consideration just $s$ times in $n$ trials, we obtain

$$\left.\begin{aligned} \sum_{r=1}^{k} x_{r-1}^{m}\,\pi\,(x_{r-1}, x_r) &\leqq \varliminf_{n=\infty} \sum_{s=0}^{n}\left(\frac{s}{n}\right)^m F(n, s)\,, \\ \sum_{r=1}^{k} x_r^m\,\pi\,(x_{r-1}, x_r) &\geqq \varlimsup_{n=\infty} \sum_{s=0}^{n}\left(\frac{s}{n}\right)^m F(n, s)\,. \end{aligned}\right\} \quad (70)$$

Consequently by the comparison of (69) and (70) we have, if $M$ denote any number intermediate between

$$\varliminf_{n=\infty} \sum_{s=0}^{n}\left(\frac{s}{n}\right)^m F(n, s) \text{ and } \varlimsup_{n=\infty} \sum_{s=0}^{n}\left(\frac{s}{n}\right)^m F(n, s)\,,$$

$$\left|\int_0^1 x^m\,\lambda\,(x)\,dx - M\right| \leqq \sum_{r=1}^{k} x_r^m\,\pi\,(x_{r-1}, x_r) - \sum_{r=1}^{k} x_{r-1}^m\,\pi\,(x_{r-1}, x_r)\,,$$

so that, if $g_k$ denote the greatest of $x_r^m - x_{r-1}^m$ $(r=1, 2, \cdots\cdots, k)$,

$$\left|\int_0^1 x^m\,\lambda\,(x)\,dx - M\right| \leqq g_k \sum_{r=1}^{k} \pi\,(x_{r-1}, x_r)\,,$$

i. e.

$$\left|\int_0^1 x^m\,\lambda\,(x)\,dx - M\right| \leqq g_k\,. \quad (71)$$

It is clear that the right-hand side expression of (71) may be made as small as we please by taking $k$ sufficiently large while the left-hand side expression is quite independent of the number $k$ of sub-intervals. It follows that

$$T_m = \lim_{n=\infty} \sum_{s=0}^{n}\left(\frac{s}{n}\right)^m F(n, s)$$

exists so as to satisfy

$$T_m = \int_0^1 x^m \lambda(x)\, dx, \qquad m \geqq 0 .$$

**12.** It has been shown that if a system of trials belongs to the generalized zero class with respect to an event such that $\pi(0, x)$ is expressible in the form (68), then the function $\lambda(x)$ must satisfy a system of integral equations

$$T_m = \int_0^1 x^m \lambda(x)\, dx , \tag{72}$$

$$m = 0, 1, 2, \cdots\cdots\cdots .$$

Now we shall conversely show that *if there exist a non-negative function $\lambda(x)$ satisfying (72) then the function $\overline{\lambda}(x) = \pi(0, x)$ can be expressed in the form*

$$\overline{\lambda}(x) = \int_0^x \lambda(x)\, dx ,$$

*and hence the system of trials belongs to the generalized zero class.*

To prove this we will first show that if $g(x)$ be any continuous function of $x$ in the interval $(0, 1)$ then

$$\lim_{n=\infty} \sum_{s=0}^{n} g\left(\frac{s}{n}\right) F(n, s) = \int_0^1 g(x)\, \lambda(x)\, dx , \tag{73}$$

in case when $\lambda(x)$ satisfies the relations (72).

If $f(x)$ be a polynomial of the form

$$f(x) = a_0 x^m + a_1 x^{m-1} + \cdots\cdots + a_m,$$

then

$$\lim_{n=\infty} \sum_{s=0}^{n} f\left(\frac{s}{n}\right) F(n, s) = a_0 \lim_{n=\infty} \sum_{s=0}^{n} \left(\frac{s}{n}\right)^m F(n, s) + a_1 \lim_{n=\infty} \sum_{s=0}^{n} \left(\frac{s}{n}\right)^{m-1} F(n, s)$$

$$+ \cdots\cdots + a_m$$

$$= a_0 T_m + a_1 T_{m-1} + \cdots\cdots + a_m ,$$

so that, if there exist a function $\lambda(x)$ satisfying (72), we have

$$\lim_{n=\infty} \sum_{s=0}^{n} f\left(\frac{s}{n}\right) F(n, s) = a_0 \int_0^1 x^m \lambda(x)\, dx + a_1 \int_0^1 x^{m-1} \lambda(x)\, dx + \cdots\cdots + a_m$$

$$= \int_0^1 f(x)\, \lambda(x)\, dx , \tag{74}$$

a result showing the validity of the relation (73) for any polynomial $f(x)$.

Now if $g(x)$ be not a polynomial then for any arbitrarily chosen positive number $\delta$, take a polynomial $f(x)$ so as to satisfy

$$|g(x)-f(x)|<\delta,$$

for all values of $x$ in the interval $(0, 1)$([1]). Then we have

$$\left|\overline{\lim_{n=\infty}}\sum_{s=0}^{n} g\left(\frac{s}{n}\right)F(n,s)-\lim_{n=\infty}\sum_{s=0}^{n} f\left(\frac{s}{n}\right)F(n,s)\right|<\delta\sum_{s=0}^{n}F(n,s)\ ([2])$$

$$<\delta,$$

and
$$\left|\int_0^1 g(x)\,\lambda(x)\,dx-\int_0^1 f(x)\,\lambda(x)\,dx\right|<\delta\int_0^1\lambda(x)\,dx$$

$$<\delta,$$

and therefore, by aid of (74),

$$\left|\overline{\lim_{n=\infty}}\sum_{s=0}^{n} g\left(\frac{s}{n}\right)F(n,s)-\int_0^1 g(x)\,\lambda(x)\,dx\right|<2\delta. \qquad (75)$$

The positive number $2\delta$ which stands on the right of (75) and of which the left-hand side expression is quite independent, may be taken as small as we please so that this relation gives rise to

$$\lim_{n=\infty}\sum_{s=0}^{n} g\left(\frac{s}{n}\right)F(n,s)=\int_0^1 g(x)\,\lambda(x)\,dx,$$

which was to be proved.

Having established the relation (73) it is now easy to prove our theorem.

Let $x_1$ be a point within the open interval $(0, 1)$ and let $g(x)$ be a continuous function of $x$ defined by

$$\begin{aligned} g(x)&=1 && \text{in the interval } (0,\ x_1-\varepsilon)\\ &=\frac{1}{\varepsilon}(x_1-x) && \text{in the interval } (x_1-\varepsilon,\ x_1)\\ &=0 && \text{in the interval } (x_1,\ 1), \end{aligned}$$

$\varepsilon$ being a positive number smaller than $x_1$. Then the relation (73) gives

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([1]) This is always possible by a well known Weierstrass' theorem.

([2]) The notation $\overline{\lim_{n=\infty}}$ means the maximum and minimum limits when $n=\infty$.

$$\overline{\lim_{n=\infty}} \sum_{0 \leqq \frac{s}{n} \leqq x_1} F(n, s) \geqq \int_0^{x_1-\epsilon} \lambda(x)\, dx\,. \tag{76}$$

Similarly by applying the relation (73) to the continuous function $g(x)$ defined by

$$\begin{aligned} g(x) &= 1 && \text{in the interval } (0, x_1) \\ &= 1 - \frac{1}{\varepsilon'}(x - x_1) && \text{in the interval } (x_1, x_1 + \varepsilon') \\ &= 0 && \text{in the interval } (x_1 + \varepsilon', 1)\,, \end{aligned}$$

we obtain

$$\overline{\lim_{n=\infty}} \sum_{0 \leqq \frac{s}{n} \leqq x_1} F(n, s) \leqq \int_0^{x_1+\varepsilon'} \lambda(x)\, dx\,. \tag{77}$$

The two relations (76) and (77) show us that $\lim\limits_{n=\infty} \sum\limits_{0 \leqq \frac{s}{n} \leqq x_1} F(n, s)$ exists and is equal to

$$\int_0^{x_1} \lambda(x)\, dx\,,$$

and therefore we have

$$\pi(0, x_1) = \int_0^{x_1} \lambda(x)\, dx\,. \tag{78}$$

The relation (78) has been obtained under the assumption $0 < x_1 < 1$. From the relation, however, it is easy to see that

$$\lim_{x_1=+0} \pi(0, x_1) = 0, \qquad \lim_{x_1=1-0} \pi(0, x_1) = 1,$$

so that all the points in the closed interval $(0, 1)$ belong to the generalized zero class. It follows therefore that the existence of a non-negative function $\lambda(x)$ satisfying (72) is sufficient to ensure the occurrence of the system of trials of the generalized zero class.

**13.** A simple case occurs when

$$\lambda(x) = \frac{1}{B(a, b)} \cdot x^{a-1}(1-x)^{b-1}, \qquad a > 0,\ b > 0,$$

where $B(a, b)$ denotes the Beta function.

In this case we have

$$T_m = \frac{1}{B(a,b)} \int_0^1 x^{m+a-1} (1-x)^{b-1} \, dx$$

$$= \frac{B(m+a, b)}{B(a,b)},$$

$$= \frac{m+a-1}{m+a+b-1} \cdot T_{m-1},$$

and consequently we get the result:

*In order that* $\overline{\lambda}(x) = \pi(0, x)$ *may be expressed in the form*

$$\overline{\lambda}(x) = \frac{1}{B(a,b)} \int_0^x x^{a-1} (1-x)^{b-1} \, dx, \qquad a>0, \ b>0 \tag{79}$$

*it is necessary and sufficient that*

$$T_m = \frac{m+a-1}{m+a+b-1} T_{m-1} \qquad (m=1, 2, 3, \cdots\cdots). \tag{80}$$

We shall now show by aid of this criterion that *the system of trials in the example given in* § 8 *when* $a=1$ *belongs to the generalized zero class.*

Referring to the notations there employed we have, by (54),

$$T_1 = p = \frac{Q}{P' + Q},$$

and by (58)

$$\frac{T_2 - T_1^2}{2} = \frac{pq}{2(1+P'+Q)},$$

so that

$$T_2 = \frac{1+Q}{1+P'+Q} \cdot T_1.$$

Now we proceed further to calculate $T_3$, $T_4$, $\cdots\cdots$.

By a quite similar method adopted for the deduction of (47) we may derive

$$V_m' = (P_m - Q_m) V'_{m-1} + (P_m - p_m) \sum_{m_1=1}^{m-2} W_{m_1, m-1}, \tag{81}$$

where

$$V_m' = \sum_{m_2=2}^{m-1} \sum_{m_1=1}^{m_2-1} \{ W_{m_1, m_2, m} - p_m \cdot W_{m_1, m_2} \}. \tag{82}$$

Since, for our case,

$$\lim_{m=\infty} m\,(1-P_m)=P', \quad \lim_{m=\infty} m\,Q_m=Q,$$

and $\dfrac{V_m'}{m^2}$ is bounded, we obtain by (81)

$$\lim_{m=\infty} \frac{m(V_m'-V_{m-1}')+(P'+Q)V_{m-1}'}{m^2}=(1-p)\lim_{m=\infty}\frac{\sum\limits_{m_1=1}^{m-2} W_{m_1,\,m-1}}{m},$$

provided that the right-hand limit exists. It is, however, the direct consequence from the existence of $\lim v_m$ in (56) that

$$\lim_{m=\infty}\frac{\sum\limits_{m_1=1}^{m-2} W_{m_1,\,m-1}}{m}$$

exists and is equal to $T_2$. It follows therefore that

$$\frac{m(V_m'-V_{m-1}')+(P'+Q)V_{m-1}'}{m^2}\to(1-p)\,T_2\,.$$

Consequently by putting

$$\frac{V_m'}{m^2}=v_m',$$

we get

$$m\,(v_m'-v_{m-1}')+(2+P'+Q)\,v_{m-1}'\to q\,T_2\,,$$

which gives, by the lemma II in § 8,

$$v_m'\to\frac{q}{2+P'+Q}\cdot T_2\,. \tag{83}$$

On the other hand we have from (82)

$$\lim_{m=\infty} v_m'=3\cdot\lim_{n=\infty}\frac{\sum\limits_{m=3}^{n} V_m'}{n^3}$$

$$=3\left[\frac{T_3}{3\,!}-\lim_{n=\infty}\frac{\sum\limits_{m=3}^{n}\left\{p_m\cdot\sum\limits_{m_2=2}^{m-1}\sum\limits_{m_1=1}^{m_2-1} W_{m_1,\,m_2}\right\}}{n^3}\right]$$

$$=3\left[\frac{T_3}{3\,!}-\lim_{n=\infty}\frac{p_n\sum\limits_{m_2=2}^{n-1}\sum\limits_{m_1=1}^{m_2-1} W_{m_1,\,m_2}}{3n^2}\right]$$

$$=3\cdot\frac{1}{3\,!}(T_3-T_1\cdot T_2)\,. \tag{84}$$

Hence equating the right-hand side expressions in (83) and (84) we arrive at

$$T_3=\frac{2+Q}{2+P'+Q}\,T_2\,.$$

Similar method may be applied to deduce the expression of $T_m$ in terms of $T_{m-1}$ and we have

$$T_m=\frac{m+Q-1}{m+P'+Q-1}T_{m-1}\,. \tag{85}$$

The relation (85) shows us that the condition (80) is satisfied if no one of $P'$ and $Q$ vanishes and hence we have the result:

*The set of games is of the generalized zero class in case when $a=1$ provided that none of $P'$ and $Q$ vanishes. The limiting probability $\pi(0,x)$ is, in this case, expressed by*

$$\pi\,(0,x)=\frac{1}{B(Q,P')}\int_0^x x^{Q-1}\,(1-x)^{P'-1}\,dx\,. \tag{86}$$

In a special case when $P'=1$ and $Q=1$, we have

$$\pi\,(0,x)=x,$$

so that the limiting probability $\pi\,(x,x+dx)$ is independent of $x$. In other cases $\pi\,(x,x+dx)$ depends always on $x$ and is proportional to

$$x^{Q-1}\,(1-x)^{P'-1}\,.$$

Especially if $Q>1$ and $P'>1$, the function $x^{Q-1}\,(1-x)^{P'-1}$ takes a maximum value when

$$x=\frac{Q-1}{Q+P'-2}\,,$$

and therefore the limiting probability $\pi(x,x+dx)$ will have the greatest value for the same value of $x$([1]).

**14.** We have, in the preceding articles, concerned with the system of trials of the generalized zero class such that the limiting probability $\pi(0,x)$ is expressible in the form (68). We shall now consider the most

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([1]) It is clear that this value of $x$ lies between 0 and 1.

general case in which the function $\pi(0, x)$ is continuous in the whole interval $(0, 1)$, but may or may not be expressed in the form (68).

If the function $\overline{\lambda}(x)=\pi(0, x)$ be continuous in the whole interval $(0, 1)$ then dividing the whole interval $(0, 1)$ into $k$ equal sub-intervals $\delta_1, \delta_2, \cdots\cdots, \delta_k$ of which $\delta_r$ has the end points $x_{r-1}$ and $x_r$ assuming

$$x_0=0, \qquad x_k=1,$$

we have

$$\overline{\lim_{n=\infty}} \sum_{s=0}^{n}\left(\frac{s}{n}\right)^m F(n, s) \leqq \sum_{r=1}^{k} x_r^m \, \pi(x_{r-1}, x_r), \tag{87}$$

$$m \geqq 0.$$

Replacing $\pi(x_{r-1}, x_r)$ in the right-hand side expression in this inequality by

$$\overline{\lambda}(x_r)-\overline{\lambda}(x_{r-1}),$$

we obtain, by virtue of

$$\overline{\lambda}(x_0)=0, \quad \text{and} \quad \overline{\lambda}(x_k)=1,$$

$$\overline{\lim_{n=\infty}} \sum_{s=0}^{n}\left(\frac{s}{n}\right)^m F(n, s) \leqq 1-\sum_{r=2}^{k}\{x_r^m-x_{r-1}^m\}\,\overline{\lambda}(x_{r-1}). \tag{88}$$

Similarly we may derive

$$\underline{\lim_{n=\infty}} \sum_{s=0}^{n}\left(\frac{s}{n}\right)^m F(n, s) \geqq x_{k-1}^m-\sum_{r=1}^{k-1}\{x_r^m-x_{r-1}^m\}\,\overline{\lambda}(x_r). \tag{89}$$

The two expressions which stand on the right of (88) and (89) respectively tend to a common limit

$$1-\int_0^1 \frac{d}{dx}(x^m)\,\overline{\lambda}(x)\,dx,$$

when the number $k$ of sub-divisions increases without limit while the left-hand side expressions are quite free from it. It follows therefore that

$$T_m=\lim_{n=\infty} \sum_{s=0}^{n}\left(\frac{s}{n}\right)^m F(n, s) \qquad (m \geqq 0)$$

exist so as to satisfy

$$T_m=1-\int_0^1 \frac{d}{dx}(x^m)\,\overline{\lambda}(x)\,dx. \tag{90}$$

We have thus shown that, if the function $\overline{\lambda}(x)$ be continuous in the interval $(0, 1)$ then it must satisfy the relation (90) or a system of integral equations

$$\frac{1-T_m}{m}=\int_0^1 x^{m-1}\,\overline{\lambda}(x)\,dx\,, \tag{91}$$

$$m=1, 2, 3, \cdots\cdots.$$

We shall again conversely show that *if there exist a continuous function $\overline{\lambda}(x)$ satisfying a system of integral equations (91) with the boundary conditions*

$$\overline{\lambda}(0)=0, \qquad \overline{\lambda}(1)=1,$$

*then the function $\overline{\lambda}(x)$ is identical with the limiting probability $\pi(0, x)$*(1) *and therefore the system of trials belongs to the generalized zero class.*

To prove this we make use of, as in § 12, a subsidiary theorem that if $g(x)$ be a continuous function of $x$ in the interval $(0, 1)$ then

$$\lim_{n=\infty}\sum_{s=0}^{n}\overline{g}\left(\frac{s}{n}\right)F(n, s)=\int_0^1 g(x)\,\overline{\lambda}(x)\,dx\,,$$

where

$$\overline{g}(x)=\int_x^1 g(x)\,dx\,.$$

If $f(x)$ be a polynomial of the form

$$f(x)=a_0\,x^{m-1}+a_1\,x^{m-2}+\cdots\cdots+a_{m-1}\,,$$

then

$$\lim_{n=\infty}\sum_{s=0}^{n}\overline{f}\left(\frac{s}{n}\right)F(n, s)=a_0\,\frac{1-T_m}{m}+a_1\,\frac{1-T_{m-1}}{m-1}+\cdots\cdots+a_{m-1}\,\frac{1-T_1}{1},$$

where

$$\overline{f}(x)=\int_x^1 f(x)\,dx\,,$$

so that if there exist a function $\overline{\lambda}(x)$ satisfying (91),

$$\lim_{n=\infty}\sum_{s=0}^{n}\overline{f}\left(\frac{s}{n}\right)F(n, s)=\int_0^1 f(x)\,\overline{\lambda}(x)\,dx\,. \tag{92}$$

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(1) For practical purposes it is sufficient to prove this theorem starting from the assumption that $\overline{\lambda}(x)$ is a non-decreasing and non-negative continuous function of $x$. In the present Part we shall however prove this theorem by the assumption here indicated.

Now if $g(x)$ be a continuous function of $x$ in the interval $(0, 1)$, let us choose a polynomial $f(x)$, corresponding to an arbitrarily chosen positive number $\delta$, so as to satisfy

$$|g(x)-f(x)|<\delta,$$

for all values of $x$ in the interval $(0, 1)$. Then evidently

$$|\overline{g}(x)-\overline{f}(x)|<\delta,$$

and therefore

$$\left|\overline{\lim_{n=\infty}}\sum_{s=0}^{n}\overline{g}\left(\frac{s}{n}\right)F(n, s)-\lim_{n=\infty}\sum_{s=0}^{n}\overline{f}\left(\frac{s}{n}\right)F(n, s)\right|<\delta. \qquad (93)$$

On the other hand if $\overline{\lambda}(x)$ satisfying (91) be continuous in the interval $(0, 1)$ then

$$\int_0^1|\overline{\lambda}(x)|\,dx$$

exists and has a finite value, say $\rho$, so that

$$\left|\int_0^1 g(x)\overline{\lambda}(x)\,dx-\int_0^1 f(x)\overline{\lambda}(x)\,dx\right|<\delta\cdot\rho. \qquad (94)$$

The two relations (93) and (94) then give, by aid of (92),

$$\lim_{n=\infty}\sum_{s=0}^{n}\overline{g}\left(\frac{s}{n}\right)F(n, s)=\int_0^1 g(x)\overline{\lambda}(x)\,dx, \qquad (95)$$

which is to be proved.

We shall now apply this subsidiary theorem to show that

$$\overline{\lambda}(x)=\pi(0, x).$$

Let $x_1$ be a point within the open interval $(0, 1)$ and let a continuous function $g(x)$ be defined as follows:

$$\begin{aligned} g(x)&=1 && \text{in the interval } (0, x_1-\varepsilon)\\ &=\frac{1}{\varepsilon}(x_1-x) && \text{in the interval } (x_1-\varepsilon, x_1)\\ &=0 && \text{in the interval } (x_1, 1), \end{aligned}$$

where $\varepsilon$ is a positive number smaller than $x_1$. Then

$$\begin{aligned} \overline{g}(x)=\int_x^1 g(x)\,dx &\leqq x_1-x \quad \text{in the interval } (0, x_1)\\ &=0 \quad \text{in the interval } (x_1, 1). \end{aligned}$$

Applying the subsidiary theorem to this function $g(x)$ we obtain

$$\varliminf_{n=\infty} \sum_{0 \leqq \frac{s}{n} \leqq x_1} \left(x_1 - \frac{s}{n}\right) F(n, s) \geqq \int_0^{x_1-\epsilon} \bar{\lambda}(x)\, dx + \int_{x_1-\epsilon}^{x_1} g(x)\, \bar{\lambda}(x)\, dx .$$

Since $0 \leqq g(x) \leqq 1$ in the interval $(x_1 - \varepsilon, x_1)$ and the function $|\bar{\lambda}(x)|$ is integrable in the same interval, the integral

$$\int_{x_1-\varepsilon}^{x_1} g(x)\, \bar{\lambda}(x)\, dx$$

will be made as small as we please by taking $\varepsilon$ sufficiently small and consequently we get

$$\varliminf_{n=\infty} \sum_{0 \leqq \frac{s}{n} \leqq x_1} \left(x_1 - \frac{s}{n}\right) F(n, s) \geqq \int_0^{x_1} \bar{\lambda}(x)\, dx . \tag{96}$$

Similarly by applying the subsidiary theorem to the continuous function $g(x)$ defined by

$$\begin{aligned} g(x) &= 1 && \text{in the interval } (0, x_1) \\ &= 1 - \frac{1}{\varepsilon'}(x - x_1) && \text{in the interval } (x_1, x_1 + \varepsilon') \\ &= 0 && \text{in the interval } (x_1 + \varepsilon', 1), \end{aligned}$$

so that

$$\begin{aligned} \bar{g}(x) &\geqq x_1 - x && \text{in the interval } (0, x_1) \\ &\geqq 0 && \text{in the interval } (x_1, 1) \end{aligned}$$

we obtain

$$\varlimsup_{n=\infty} \sum_{0 \leqq \frac{s}{n} \leqq x_1} \left(x_1 - \frac{s}{n}\right) F(n, s) \leqq \int_0^{x_1} \bar{\lambda}(x)\, dx . \tag{97}$$

This relation, combined with (96), gives us

$$\lim_{n=\infty} \sum_{0 \leqq \frac{s}{n} \leqq x_1} \left(x_1 - \frac{s}{n}\right) F(n, s) = \int_0^{x_1} \bar{\lambda}(x)\, dx . \tag{98}$$

Consequently by taking another point $x_2$ in the open interval $(0, 1)$ such that $x_1 < x_2$ we have

$$\lim_{n=\infty} \sum_{0 \leqq \frac{s}{n} \leqq x_2} \left(x_2 - \frac{s}{n}\right) F(n, s) = \int_0^{x_2} \bar{\lambda}(x)\, dx\,, \tag{99}$$

and from the two relations (98) and (99) we get by subtraction

$$\lim_{n=\infty} \left\{ (x_2 - x_1) \sum_{0 \leqq \frac{s}{n} \leqq x_1} F(n, s) + \sum_{x_1 < \frac{s}{n} \leqq x_2} \left(x_2 - \frac{s}{n}\right) F(n, s) \right\}$$

$$= \int_{x_1}^{x_2} \bar{\lambda}(x)\, dx\,.$$

Accordingly we may derive

$$(x_2 - x_1) \overline{\lim_{n=\infty}} \sum_{0 \leqq \frac{s}{n} \leqq x_1} F(n, s) \leqq \int_{x_1}^{x_2} \bar{\lambda}(x)\, dx$$

$$\leqq (x_2 - x_1) \underline{\lim_{n=\infty}} \sum_{0 \leqq \frac{s}{n} \leqq x_2} F(n, s)\,,$$

or

$$\overline{\lim_{n=\infty}} \sum_{0 \leqq \frac{s}{n} \leqq x_1} F(n, s) \leqq \frac{1}{x_2 - x_1} \int_{x_1}^{x_2} \bar{\lambda}(x)\, dx\,, \tag{100}$$

$$\underline{\lim_{n=\infty}} \sum_{0 \leqq \frac{s}{n} \leqq x_2} F(n, s) \geqq \frac{1}{x_2 - x_1} \int_{x_1}^{x_2} \bar{\lambda}(x)\, dx. \tag{101}$$

The left-hand side expression of (100) is independent of $x_2$ while the right-hand side expression tends to $\bar{\lambda}(x_1)$ when $x_2$ converges to $x_1$. It follows therefore that

$$\overline{\lim_{n=\infty}} \sum_{0 \leqq \frac{s}{n} \leqq x_1} F(n, s) \leqq \bar{\lambda}(x_1)\,. \tag{102}$$

Similarly from (101)

$$\underline{\lim_{n=\infty}} \sum_{0 \leqq \frac{s}{n} \leqq x_2} F(n, s) \geqq \bar{\lambda}(x_2)\,. \tag{103}$$

The two relations (102) and (103) show us that

$$\lim_{n=\infty} \sum_{0 \leqq \frac{s}{n} \leqq x} F(n, s) = \pi(0, x)$$

exists and is equal to $\bar{\lambda}(x)$ if $0<x<1$. But by the assumption

$$\bar{\lambda}(0)=0, \qquad \bar{\lambda}(1)=1,$$

we have

$$\lim_{x=+0} \pi(0, x)=0, \qquad \lim_{x=1-0} \pi(0, x)=1,$$

and therefore for all values of $x$ in the closed interval $(0, 1)$ we get

$$\pi(0, x)=\bar{\lambda}(x),$$

thus establishing the theorem completely ([1]).

Combining the results obtained above we have therefore

*In order that a system of trials may be of the generalised zero class it is necessary and sufficient that there should exist a continuous function $\bar{\lambda}(x)$ satisfying the system of integral equations*

$$\frac{1-T_m}{m}=\int_0^1 x^{m-1}\bar{\lambda}(x)\,dx. \tag{91}$$

$$m=1, 2, \cdots\cdots$$

*with the boundary conditions*

$$\bar{\lambda}(0)=0, \qquad \bar{\lambda}(1)=1, \tag{104}$$

*and if there exist such a continuous function $\bar{\lambda}(x)$ then*

$$\pi(0, x)=\bar{\lambda}(x).$$

That the boundary conditions (104) are necessary, i. e. they do not follow from the existence of a continuous function $\bar{\lambda}(x)$ satisfying (91) can be shown by an example when

$$T_m=1 \quad (m=0, 1, 2, \cdots\cdots).$$

The system of trials, in this case, belongs to the first class considered at Part I and the system of integral equations (91) has a continuous solution $\bar{\lambda}(x)=0$, not satisfying one of the boundary conditions (104).

In the case when $D\bar{\lambda}(x)$ which is any one of the four derivatives of $\bar{\lambda}(x)$, is limited in the interval such that $D\bar{\lambda}(x)$ is integrable, or has at most points of infinite discontinuity forming a closed enumerable set

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([1]) It has indirectly been shown that $\bar{\lambda}(x)$ is a non-negative and non-decreasing function of $x$. Also it has been shown that every point in the open interval $(0, 1)$ belongs to the generalized zero class even in the case when the boundary conditions (104) are not satisfied.

such that $D\bar{\lambda}(x)$ has a convergent improper integral, the integration by parts([1]) may be applied to the integral standing on the right of (91 and we have, by aid of (104),

$$T'_m = \int_0^1 x^m \lambda(x)\, dx, \quad m = 0, 1, 2, \cdots\cdots,$$

where

$$\lambda(x) = D\bar{\lambda}(x),$$

which is the system of integral equations considered at § 11.

**15.** By means of the relation (91) which the function

$$\bar{\lambda}(x) = \pi(0, x)$$

satisfies for a system of trials of the generalized zero class it is easy to derive an analytical expression of $\bar{\lambda}(x)$.

Since $\bar{\lambda}(x)$ is a continuous function of $x$ in the interval $(0, 1)$ we have by Fourier's theorem

$$\bar{\lambda}(x) = \frac{1}{\pi}\int_0^\infty dy \int_0^1 \bar{\lambda}(x') \cos y(x'-x)\, dx',$$

for values of $x$ in the open interval $(0, 1)$. Now by putting

$$\phi(x, y) = \int_0^1 \bar{\lambda}(x') \cos y(x'-x)\, dx',$$

so that

$$\bar{\lambda}(x) = \frac{1}{\pi}\int_0^\infty \phi(x, y)\, dy, \tag{105}$$

we get

$$\phi(x, y) = \cos xy \int_0^1 \left\{1 - \frac{(yx')^2}{2!} + \frac{(yx')^4}{4!} - \cdots\cdots\right\} \bar{\lambda}(x')\, dx'$$

$$+ \sin xy \int_0^1 \left\{\frac{(yx')}{1!} - \frac{(yx')^3}{3!} + \cdots\cdots\right\} \bar{\lambda}(x')\, dx'.$$

The two series under the integral signs on the right-hand side are of course uniformly convergent in the interval $(0 \leqq x' \leqq 1)$ and they can be integrated term by term. It follows therefore, by aid of (91),

$$\phi(x, y) = \left\{\frac{T'_1}{1!} - \frac{T'_3}{3!} y^2 + \frac{T'_5}{5!} y^4 - \cdots\cdots\right\} \cos xy$$

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([1]) Hobson, Theory of Functions of a real variable (1907), p. 409.

$$+\left\{\frac{T_2'}{2\,!}y-\frac{T_4'}{4\,!}y^3+\frac{T_6'}{6\,!}y^5-\cdots\cdots\right\}\sin xy,$$

where

$$T_m'=1-T_m,$$

and consequently, by substituting this expression of $\phi(x, y)$ into (105) we get

$$\bar{\lambda}(x)=\frac{1}{\pi}\int_0^\infty\left[\left\{\sum_{m=0}^{\infty}(-1)^m\frac{T_{2m+1}'}{(2m+1)\,!}y^{2m}\right\}\cos xy\right.$$

$$\left.+\left\{\sum_{m=0}^{\infty}(-1)^{m+1}\frac{T_{2m}'}{(2m)\,!}y^{2m-1}\right\}\sin xy\right]dy\,(^1). \qquad (106)$$

Observe that this expression of $\bar{\lambda}(x)$ is identical with that obtained in § 9 for the system of trials of the first class.

## Part III.

### Determinate System of Trials with respect to an Event.

**16.** We now go on to the consideration of a more general system of trials of which two special kinds have been treated in the Parts I and II.

The limiting probability $\pi(\alpha, \beta)$ being defined, as before, as the limit when $n=\infty$ of the probability that the number $s$ of the happening of an event in $n$ trials will satisfy

$$\alpha \leqq \frac{s}{n} \leqq \beta,$$

a point $x$ in the interval $(0, 1)$ may be such that $\pi(x-\varepsilon, x+\varepsilon')$ exists at least for points $\varepsilon$ and $\varepsilon'$ belonging to two positive sets $S$ and $S'$ respectively which have the point 0 as their common limiting point. For such a point $x$, if

$$\overline{\lim_{n=\infty}}\,\pi_n(x-\varepsilon_1, x+\varepsilon_1') \quad \text{and} \quad \underline{\lim_{n=\infty}}\,\pi_n(x-\varepsilon_1, x+\varepsilon_1'),$$

both of which necessarily exist for *all* positive $\varepsilon_1$ and $\varepsilon_1'$ and also are monotonic functions of them, be denoted by

(1) That the value of this expression is 0 when $x=0$ and is $^1/_2$ when $x=1$ is also at once seen from Fourier's theorem.

$$\overline{\pi}(x-\varepsilon_1, x+\varepsilon_1') \quad \text{and} \quad \underline{\pi}(x-\varepsilon_1, x+\varepsilon_1'), \text{ (}^1\text{)}$$

respectively, we have easily

$$\lim_{\epsilon_1, \epsilon_1'=+0} \overline{\pi}(x-\varepsilon_1, x+\varepsilon_1') = \lim_{\epsilon_1, \epsilon_1'=+0} \underline{\pi}(x-\varepsilon_1, x+\varepsilon_1') = \lambda,$$

where $\lambda$ is a definite number between 0 and 1 and therefore

$$\lim_{\epsilon, \epsilon'=+0} \pi(x-\varepsilon, x+\varepsilon') = \lambda,$$

independent of any particular selection of two positive sequences $(\varepsilon)$ and $(\varepsilon')$ *belonging to* $S$ *and* $S'$ respectively and converging to the limit zero.

Let us call such a point $x$ the *point at which the limiting probability* $\pi(x-0, x+0)$ *is determinate and is equal to* $\lambda$, and let the two sets of points at which

$$\lambda=0 \quad \text{and} \quad \lambda \neq 0$$

respectively be denoted by $B_0$ and $B_1$. It is evident that the set $A_1$ defined in § 1 is a component of the set $B_1$ and the set $A_0'$ defined in § 10 is a component of the set $B_0$.

If now $x$ be a point of the set $B_0$ it is easy to see that

$$\pi(x, x)=0.$$

Also in this case it can be shown that, if $\varepsilon'$ belongs to the set $S'$ then

$$\overline{\pi}(x, x+\varepsilon') = \underline{\pi}(x, x+\varepsilon')$$

so that $\pi(x, x+\varepsilon')$ exists.

For if a positive number $\delta$ belonging to $S$ and another positive number $\delta'$ belonging to $S'$ such that $\delta' < \varepsilon'$ be taken we have

$$\overline{\pi}(x, x+\varepsilon') \leqq \pi(x-\delta, x+\varepsilon'),$$

$$\underline{\pi}(x, x+\varepsilon') \geqq \pi(x-\delta, x+\varepsilon') - \pi(x-\delta, x+\delta'),$$

and therefore by subtraction

$$\overline{\pi}(x, x+\varepsilon') - \underline{\pi}(x, x+\varepsilon') \leqq \pi(x-\delta, x+\delta'),$$

of which the right-hand side expression may be made as small as we please by taking $\delta$ and $\delta'$ sufficiently small. Consequently we have

(1) These two limits are of course equal to each other if $\epsilon_1$ belongs to the set $S$ and $\epsilon_1'$ to the set $S'$.

$$\overline{\pi}(x, x+\varepsilon') = \underline{\pi}(x, x+\varepsilon'). \tag{107}$$

Similarly we may show that, if $\varepsilon$ belongs to the set $S$

$$\overline{\pi}(x-\varepsilon, x) = \underline{\pi}(x-\varepsilon, x).$$

Again if $x$ be a point of the set $B_1$ having the limiting probability $\pi(x-0, x+0) = \lambda$, then for a point $\varepsilon'$ belonging to the set $S'$

$$\overline{\pi}(x-0, x+\varepsilon') = \underline{\pi}(x-0, x+\varepsilon')(^1), \tag{108}$$

and for a point $\varepsilon$ belonging to the set $S$

$$\overline{\pi}(x-\varepsilon, x+0) = \underline{\pi}(x-\varepsilon, x+0).$$

These results, combined with

$$\pi(x-0, x+0) = \lambda,$$

give also

$$\overline{\pi}(x+0, x+\varepsilon') = \underline{\pi}(x+0, x+\varepsilon'),$$
$$\overline{\pi}(x-\varepsilon, x-0) = \underline{\pi}(x-\varepsilon, x-0).$$

Now let us show that

*If all points within the closed interval* $(\alpha \leqq x \leqq \beta)$ *belong to the set* $B_0$ *then* $\pi(\alpha, x)$ *exists and is a continuous function of* $x$ *within the same closed interval.*

That the function $\overline{\pi}(\alpha, x)$ which is a maximum limit of $\pi_n(\alpha, x)$ when $n=\infty$ and hence exists for all values of $x$ in the interval $(\alpha, \beta)$, is a continuous function of $x$ is at once seen. For since it is a monotonic function of $x$, if it becomes discontinuous, say at $x=x_1$, we must have

$$\left.\begin{aligned} \overline{\pi}(\alpha, x_1-0) &= k_1(^2), \\ \overline{\pi}(\alpha, x_1+0) &= k_2, \\ k_1 &< k_2. \end{aligned}\right\} \tag{109}$$

On the other hand, as $x_1$ belongs to the set $B_0$, we may take two positive numbers $\varepsilon$ and $\varepsilon'$ such that

$$\pi(x_1-\varepsilon, x_1+\varepsilon') < k_2 - k_1. \tag{109$'$}$$

Then we have from (109) and (109)$'$

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(1) $\overline{\pi}(x-0, x+\epsilon')$ and similar notations denote $\lim\limits_{\epsilon_1=+0} \overline{\pi}(x-\epsilon_1, x+\epsilon')$, etc.

(2) When $x_1$ coincides with $\alpha$ this relation is to be replaced by

$$\overline{\pi}(\alpha, \alpha) = 0.$$

$$\overline{\pi}\,(\alpha, x_1-0)+\pi\,(x_1-\varepsilon,\ x_1+\varepsilon')<\overline{\pi}\,(\alpha, x_1+0),$$

which is certainly impossible.

Just in the same manner we may show that the function $\underline{\pi}\,(\alpha, x)$ which is a minimun limit of $\pi_n(\alpha, x)$ is continuous in the interval $(\alpha, \beta)$.

Having shown that $\overline{\pi}\,(\alpha, x)$ and $\underline{\pi}\,(\alpha, x)$ are continuous functions of $x$ in the interval $(\alpha, \beta)$, let us consider the function

$$M(x)=\overline{\pi}\,(\alpha, x)-\underline{\pi}\,(\alpha, x).$$

Since this function is continuous in the interval $(\alpha, \beta)$ and becomes zero at $x=\alpha$, the zero points of the function must form a closed set, say $g$, which consists at least of one point. Let any one of the complementary intervals to $g$ in $(\alpha, \beta)$, if exist, be $x_1x_2$ of which $x_1$ is the end point on the left. Then evidently, as $x_1$ belongs to the set $g$,

$$M(x_1)=0,$$

and by choosing a proper positive number $\varepsilon'$ smaller than $x_2-x_1$ such that

$$\overline{\pi}\,(x_1, x_1+\varepsilon')=\underline{\pi}\,(x_1, x_1+\varepsilon')$$

which is always possible as $x_1$ belongs to the set $B_0$, we have

$$M(x_1+\varepsilon')=0.$$

But this contradicts the supposition that $x_1+\varepsilon'$ belongs to the internal points of one of the complementary intervals to $g$. It follows therefore that there exists no complementary interval to $g$ and hence we have

$$M(x)=0,$$

i.e.

$$\overline{\pi}\,(\alpha, x)=\underline{\pi}\,(\alpha, x),$$

throughout the whole interval $(\alpha, \beta)$ — a result showing that $\pi(\alpha, x)$ exists and is continuous in the same interval.

From this theorem, combined with the result obtained in § 10, we have:

*If all points within the closed interval $(\alpha, \beta)$ belong to the set $B_0$ then every point within the open interval $(\alpha, \beta)$ is a point of the set $A_0'$ defined in* § 10.

This shows that in a sufficiently small neighbourhood of a point belonging to $B_0$ but not to $A_0'$ there exist an infinite number of points not belonging to $B_0$ and therefore

*If $B_0'$ be the set of points complementary to $A_0'$ in the set $B_0$, then every point of the set $B_0'$ is a limiting point of the set $C(B_0)$ which is a complementary set of $B_0$ in the continuum* $(0, 1)$.

As a special case we get:

If all points within the closed interval $(0, 1)$ belong to the set $(B_0, B_1)$ where $(B_0, B_1)$ denotes the sum set of $B_0$ and $B_1$, then the set complementary to $A_0'$ in $(0, 1)$ is identical with the set $B_1$ together with its limiting points.

Further let us show that

*If all points within the closed interval $(\alpha, \beta)$ of which $\alpha$ is the point of the set $B_0$ belong to the set $(B_0, B_1)$ where $(B_0, B_1)$ denotes the sum set of $B_0$ and $B_1$ then the two functions $\overline{\pi}(\alpha, x)$ and $\underline{\pi}(\alpha, x)$ are continuous and identical to each other at points of the set $B_0$ in the interval $(\alpha, \beta)$.*

It can be shown in a quite similar manner as in the above that the two functions $\overline{\pi}(\alpha, x)$ and $\underline{\pi}(\alpha, x)$ both of which are monotonic, are continuous at points of the set $B_0$ in the interval $(\alpha, \beta)$.

Now let $x_1$ be a point of the set $B_0$ in $(\alpha, \beta)$. If, in the immediate neighbourhood of the point $x_1$, there exists no point at which

$$\overline{\pi}(\alpha, x) = \underline{\pi}(\alpha, x),$$

then let $AB$ be the greatest interval including $x_1$ within which we have

$$\overline{\pi}(\alpha, x) \neq \underline{\pi}(\alpha, x).$$

The end point $x_1'$, on the left of the interval, which necessarily exists, as at least at $x=\alpha$ we have $\overline{\pi}(\alpha, x) = \underline{\pi}(\alpha, x)$, must be a limiting point of a set of points for which

$$\overline{\pi}(\alpha, x) = \underline{\pi}(\alpha, x),$$

so that we obtain

$$\overline{\pi}(\alpha, x_1' - 0) = \underline{\pi}(\alpha, x_1' - 0).$$

Since $x_1'$ belongs to either the set $B_0$ or the set $B_1$, by choosing a proper positive number $\varepsilon'$ smaller than $x_1 - x_1'$ such that

$$\overline{\pi}(x_1' - 0, x_1' + \varepsilon') = \underline{\pi}(x_1' - 0, x_1' + \varepsilon'),$$

we have

$$\overline{\pi}(\alpha, x_1' + \varepsilon') = \underline{\pi}(\alpha, x_1' + \varepsilon'),$$

which is however contradictory to that $x_1' + \varepsilon'$ belongs to the interna points of the interval $AB$.

This shows that every point of the set $B_0$ is a limiting point of the set of points at which

$$\overline{\pi}(\alpha, x) = \underline{\pi}(\alpha, x),$$

and therefore from the continuity of the two functions at points of the set $B_0$ we shall have, if $x_1$ belongs to the set $B_0$,

$$\overline{\pi}(\alpha, x_1) = \underline{\pi}(\alpha, x_1),$$

thus establishing the theorem completely.

It will easily be seen that the set $B_1$ is an enumerable set so that the complementary set to $B_1$ in any interval is an everywhere dense set in that interval([1]). For if $\delta$ be an arbitrarily chosen positive number there may be at most a finite number of points of the set $B_1$ at which the limiting probability $\pi(x-0, x+0)$ exceeds this number $\delta$.

Then it follows readily that the two functions $\overline{\pi}(\alpha, x)$ and $\underline{\pi}(\alpha, x)$ have the same saltus different from zero at points of the enumerable set $B_1$ in the interval $(\alpha, \beta)$.

Consequently *in the case when all points in the interval* $(0, 1)$ *belong to the set* $(B_0, B_1)$ *the function* $\pi(0, x)$ *exists and is continuous in the domain consisting of points of the set* $B_0$, *while it is indeterminate or at least becomes discontinuous at points of the set* $B_1$.

That conversely if the function $\pi(0, x)$ exists and is continuous in the domain consisting of an every-where dense set $E$ of points in the interval $(0, 1)$ then every point of the whole interval belongs either to the set $B_0$ or to the set $B_1$, the set $E$ being a component of $B_0$, can at once be shown. Such a system of trials is called the *determinate system* with respect to the event under consideration.

The conditions for the occurrence of such a case as well as the determination of the function $\pi(0, x)$ in this case will be fully discussed in the present Part.

**17.** We now first show that *the existence of all* $T_m$ *is the necessary and sufficient condition for the occurrence of a determinate system with respect to an event.*

If every point in the interval $(0, 1)$ belong to either the set $B_0$ or the set $B_1$ then it has been shown that $\overline{\pi}(0, x)$ and $\underline{\pi}(0, x)$ are continuous and identical with each other in the domain consisting of points of the set $B_0$ which is everywhere dense in the interval $(0, 1)$, while

([1]) Of course the complementary set to $B_1$ has the measure equal to the length of the interval.

they have the same saltus at points of the set $B_1$. Thence if we define $\bar{\lambda}(x)$ by

$$\left.\begin{aligned}\bar{\lambda}(x)&=\pi(0,x) \text{ at points of the set } B_0\,(^1),\\ &=\text{any intermediate number between } \overline{\pi}(0,x) \text{ and } \underline{\pi}(0,x)\\ &\quad\text{at points of the set } B_1 \text{ in the open interval } (0,1),\end{aligned}\right\}\tag{110}$$

then the function $\bar{\lambda}(x)$ will be a determinate monotonic function of $x$ in the whole interval $(0, 1)$, so that the integral

$$\int_0^1 \frac{d}{dx}(x^m)\,.\,\bar{\lambda}(x)\,dx, \qquad m\geqq 0,$$

always exists.

Now divide the interval $(0, 1)$ into $k$ sub-intervals $\delta_1, \delta_2, \cdots\cdots, \delta_k$ of which $\delta_r$ has the end points $X_{r-1}$ and $X_r$ both belonging to the set $B_0$ at least except the end points of the interval $(0, 1)$. Then we have

$$\overline{\lim_{n=\infty}} \sum_{s=0}^{n}\left(\frac{s}{n}\right)^m F(n,s)\leqq \sum_{r=1}^{k} X_r^m\,\pi(X_{r-1}, X_r).$$

Since $X_r\,(r=1, 2, \cdots\cdots, k-1)$ belong to the set $B_0$,

$$\pi(X_{r-1}, X_r)=\bar{\lambda}(X_r)-\bar{\lambda}(X_{r-1}), \qquad (r=1, 2, \cdots\cdots, k)$$

where $$\bar{\lambda}(X_0)=0, \qquad \bar{\lambda}(X_k)=1,$$

and therefore

$$\overline{\lim_{n=\infty}} \sum_{s=0}^{n}\left(\frac{s}{n}\right)^m F(n,s)\leqq 1-\sum_{r=2}^{k}\{X_r^m-X_{r-1}^m\}\,\bar{\lambda}(X_{r-1}).\tag{111}$$

Similarly

$$\underline{\lim_{n=\infty}} \sum_{s=0}^{n}\left(\frac{s}{n}\right)^m F(n,s)\geqq X_{k-1}^m-\sum_{r=1}^{k-1}\{X_r^m-X_{r-1}^m\}\,\bar{\lambda}(X_r).\tag{111$'$}$$

The two expressions standing on the right of (111) and (111)′ respectively tend to the common limit

$$1-\int_0^1 \frac{d}{dx}(x^m)\,.\,\bar{\lambda}(x)\,dx,$$

when the sub-division advances without limit such that the greatest sub-interval of the sub-division has the limit zero, and hence we have

---

(1) $\overline{\lambda}(0)=0$ and $\overline{\lambda}(1)=1$ are assumed even if the point $x=0$ or $x=1$ belong to the set $B_1$; i. e. in such a case $\overline{\lambda}(0)$ and $\overline{\lambda}(1)$ are to be defined by $\overline{\lambda}(0)=\lambda(-0)$, $\overline{\lambda}(1)=\overline{\lambda}(1+0)$.

$$T_m = 1 - \int_0^1 \frac{d}{dx}(x^m) . \bar{\lambda}(x)\, dx,$$

a result showing the existence of $T_m$ in case when the system of trials belongs to the determinate system.

Conversely if there exist *a non-negative monotonic function* $\bar{\lambda}(x)$ satisfying

$$T_m = 1 - \int_0^1 \frac{d}{dx}(x^m) . \bar{\lambda}(x)\, dx, \qquad (112)$$

$$m = 1, 2, 3, \cdots\cdots$$

then it can be shown in a quite similar manner as in § 14 that

$$\bar{\lambda}(x) = \pi(0, x)(^1),$$

at points of continuity of $\bar{\lambda}(x)$ forming an everywhere dense set in the open interval $(0, 1)$ so that the system of trials is the determinate system.

Consequently in order to show that the existence of $T_m$ $(m = 1, 2, \cdots\cdots)$ is a sufficient condition for the occurrence of a determinate system it is sufficient to prove that there always exists a non-negative monotonic function $\bar{\lambda}(x)$ satisfying the system of integral equations (112).

Now let the whole interval $(0, 1)$ be divided into $k_1$ sub-intervals $\delta_1^{(1)}, \delta_2^{(1)}, \cdots\cdots, \delta_{k_1}^{(1)}$ of which $\delta_r^{(1)}$ has the end points $X_{r-1}^{(1)}$ and $X_r^{(1)}$ and let the sequence $(n_1)$ be so chosen from the sequence of natural numbers $(1, 2, 3, \cdots\cdots)$ that all of

$$\pi_{n_1}(X_{r-1}^{(1)}, X_r^{(1)}) - \pi_{n_1}(X_{r-1}^{(1)}, X_{r-1}^{(1)})(^2),$$

$$r = 1, 2, 3, \cdots\cdots, k_1$$

tend to determinate limits $\pi^{(1)}(X_{r-1}^{(1)}, X_r^{(1)})$ when $n_1$ becomes infinite respectively. Then we have

$$\sum_{r=1}^{k_1} \pi^{(1)}(X_{r-1}^{(1)}, X_r^{(1)}) = 1,$$

and

$$\sum_{r=1}^{k_1} (X_{r-1}^{(1)})^m \pi^{(1)}(X_{r-1}^{(1)}), (X_r^{(1)}) \leqq T_m \leqq \sum_{r=1}^{k_1} (X_r^{(1)})^m \pi^{(1)}(X_{r-1}^{(1)}, X_r^{(1)}),$$

$$m = 1, 2, 3, \cdots\cdots$$

---

(1) We need not here the boundary conditions $\lim_{x=+0} \overline{\lambda}(x) = 0$, $\lim_{x=1-0} \overline{\lambda}(x) = 1$. If one or both of them be not satisfied, the point 0 or 1 or both belong to the set $B_1$.

(2) $\pi_{n_1}(0, 0)$ is to be replaced by 0.

provided that $T_m$ exist, and consequently by putting

$$\bar{\lambda}^{(1)}(X_r^{(1)}) = \sum_{r=1}^{r} \pi^{(1)}(X_{r-1}^{(1)}, X_r^{(1)}),$$
$$r=1, 2, \cdots\cdots, k_1,$$

so that $\bar{\lambda}^{(1)}(x)$ has a determinate value at points $x=X_r^{(1)}$ forming a finite set $G_1$, we get

$$\sum_{r=1}^{k_1} (X_{r-1}^{(1)})^m \{\bar{\lambda}^{(1)}(X_r^{(1)}) - \bar{\lambda}^{(1)}(X_{r-1}^{(1)})\} \leqq T_m \text{ (}^1\text{)},$$

$$\sum_{r=1}^{k_1} (X_r^{(1)})^m \{\bar{\lambda}(X_r^{(1)}) - \bar{\lambda}^{(1)}(X_{r-1}^{(1)})\} \geqq T_m,$$

i.e.

$$(X_{k_1-1}^{(1)})^m - \sum_{r=1}^{k_1-1} \{(X_r^{(1)})^m - (X_{r-1}^{(1)})^m\} \bar{\lambda}^{(1)}(X_r^{(1)}) \leqq T_m,$$

$$1 - \sum_{r=2}^{k_1} \{(X_r^{(1)})^m - (X_{r-1}^{(1)})^m\} \bar{\lambda}^{(1)}(X_{r-1}^{(1)}) \geqq T_m.$$

Again let the sub-intervals $\delta_1^{(1)}, \delta_2^{(1)}, \cdots\cdots, \delta_k^{(1)}$ be further sub-divided so that the whole interval $(0, 1)$ consists of $k_2$ sub-intervals $\delta_1^{(2)}, \delta_2^{(2)}, \cdots\cdots, \delta_{k_2}^{(2)}$ of which $\delta_r^{(2)}$ has the end points $X_{r-1}^{(2)}$ and $X_r^{(2)}$ and let the sequence $(n_2)$ be so chosen from the sequence $(n_1)$ that all of

$$\pi_{n_2}(X_{r-1}^{(2)}, X_r^{(2)}) - \pi_{n_2}(X_{r-1}^{(2)}, X_{r-1}^{(2)})$$

tend to determinate limits $\pi^{(2)}(X_{r-1}^{(2)}, X_r^{(2)})$ when $n_2$ becomes infinite. Then by putting

$$\bar{\lambda}^{(2)}(X_r^{(2)}) = \sum_{r=1}^{r} \pi^{(2)}(X_{r-1}^{(2)}, X_r^{(2)}),$$

we obtain a function $\bar{\lambda}^{(2)}(x)$ defined at points of a finite set $G_2$ which contains $G_1$ as its component set. The function $\bar{\lambda}^{(2)}(x)$ clearly satisfies

$$\bar{\lambda}^{(2)}(X_r^{(1)}) = \bar{\lambda}^{(1)}(X_r^{(1)}), \quad r=1, 2, \cdots\cdots, k_1,$$

and

$$(X_{k_2-1}^{(2)})^m - \sum_{r=1}^{k_2-1} \{(X_r^{(2)})^m - (X_{r-1}^{(2)})^m\} \bar{\lambda}^{(2)}(X_r^{(2)}) \leqq T_m,$$

$$1 - \sum_{r=2}^{k_2} \{(X_r^{(2)})^m - (X_{r-1}^{(2)})^m\} \bar{\lambda}^{(2)}(X_{r-1}^{(2)}) \geqq T_m.$$

---

(1) $\bar{\lambda}^{(1)}(0)$ is to be always replaced by zero.

Let further sub-divisions on these sub-intervals be made, and so on indefinitely, so that at the $l$th stage of the process we have a function $\bar{\lambda}^{(l)}(x)$, defined at points of a finite set $G_l$ which contains $G_{l-1}$ as its component, so as to satisfy

$$\bar{\lambda}^{(l)}(x)=\bar{\lambda}^{(l-1)}(x),$$

at a point $x$ belonging to $G_{l-1}$, and also

$$(X_{k_l-1}^{(l)})^m-\sum_{r=1}^{k_l-1}\{(X_r^{(l)})^m-(X_{r-1}^{(l)})^m\}\bar{\lambda}^{(l)}(X_r^{(l)})\leqq T_m, \qquad (113)$$

$$1-\sum_{r=2}^{k_l}\{(X_r^{(l)})^m-(X_{r-1}^{(l)})^m\}\bar{\lambda}^{(l)}(X_{r-1}^{(l)})\geqq T_m. \qquad (113)'$$

Now let the system of sub-divisions be so constructed as to form a convergent system of sub-divisions([1]), then the limiting set $G_\omega$ of a sequence of sets $G_1, G_2, \cdots\cdots$ so formed of which each set contains the preceding one as a component, will be an everywhere dense set in the interval $(0, 1)$, and as the sequence $(n_l)$ is a component of $(n_{l-1})$ we have at points $x=X_r^{(l)}$ belonging to $G_l$ and therefore also to $G_\omega$,

$$\bar{\lambda}^{(l)}(x)=\bar{\lambda}^{(l+1)}(x)=\bar{\lambda}^{(l+2)}(x)=\cdots\cdots.$$

Since, in this case, the difference between the two expressions which stand on the left of (113) and (113)′ will have the limit zero when $l=\infty$, we get

$$\lim_{l=\infty}[1-\sum_{r=2}^{k_l}\{(X_r^{(l)})^m-(X_{r-1}^{(l)})^m\}\bar{\lambda}^{(l)}(X_{r-1}^{(l)})]=T_m,$$

and therefore if $\bar{\lambda}^{(\omega)}(x)$([2]) denote the limiting function of $\bar{\lambda}^{(1)}(x), \bar{\lambda}^{(2)}(x), \cdots\cdots$ we have

$$\lim_{l=\infty}[1-\sum_{r=2}^{k_l}\{(X_r^{(l)})^m-(X_{r-1}^{(l)})^m\}\bar{\lambda}^{(\omega)}(X_{r-1}^{(l)})=T_m. \qquad (114)$$

Now $\bar{\lambda}^{(\omega)}(x)$ is a non-negative monotonic function defined at points of an everywhere dense set in the interval $(0, 1)$. Consequently if $\bar{\lambda}(x)$ denote the extended function of $\bar{\lambda}^{(\omega)}(x)$ in the extended domain which

---

([1]) I. e. the greatest sub-interval at the $l$th stage has the limit zero when $l=\infty$.

([2]) $\bar{\lambda}^{(\omega)}(x)$ is a function defined at points of the set $G_\omega$ by

$$\bar{\lambda}^{(\omega)}(x)=\bar{\lambda}^{(l)}(x)=\bar{\lambda}^{(l+1)}(x)=\cdots\cdot$$

if $x$ belongs to the set $G_l$ and consequently also to all subsequent sets of the sequence.

coincides with the entire interval $(0, 1)$ then $\bar{\lambda}(x)$ is a non-negative monotonic function of $x$ in the whole interval $(0, 1)$ and therefore the function $\frac{d}{dx}(x^m)\,\bar{\lambda}(x)$ will be an integrable function in the same interval. The relation (114) then gives

$$1-\int_0^1 \frac{d}{dx}(x^m).\bar{\lambda}(x)\,dx=T_m.$$

Thus it has been shown that if all $T_m\,(m=1, 2, 3, \cdots\cdots)$ exist then there always exists a non-negative monotonic function $\bar{\lambda}(x)$ satisfying a system of integral equations (112).

Combining the results we have therefore:

*In order that a system of trials may be a determinate system with respect to an event it is necessary and sufficient that all $T_m$ $(m=1, 2, 3, \cdots\cdots)$ should exist.*

*If all $T_m$ exist, then the system of integral equations (112) has always a solution $\bar{\lambda}(x)$ which is monotonic in the interval $(0, 1)$. The set of points of continuity of the function $\bar{\lambda}(x)$ thus found is identical with the set $B_0$ and the set of points of discontinuity of the function, with the set $B_1$.*

**18.** We shall now transform the system of integral equations (112) in a somewhat different form.

If $\bar{\lambda}(x)$ defined by (112) be a monotonic function of $x$ which is continuous at points of the set $B_0$ and becomes discontinuous at points of the set $B_1$, then the function $\psi(x)$ defined, at points $x$ of the set $B_0$, by

$$\psi(x)=\bar{\lambda}(x)-w(x),$$

where $w(x)$ denotes the sum or the limiting sum of the saltuses of the function $\bar{\lambda}(x)$ in the interval $(0, x)$ will be continuous in the domain consisting of points of the set $B_0$ everywhere dense in the interval $(0, 1)$. Let $\Psi(x)$ and $W(x)$ be the extended functions of $\psi(x)$ and $w(x)$ from their values at points of the set $B_0$ in the extended domain which coincides with the entire interval $(0, 1)$. Then $\Psi(x)$ is a monotonic and continuous function of $x$ in the whole interval and satisfies, at every point of the set $B_0$,

$$\Psi(x)=\bar{\lambda}(x)-W(x),$$

---

(1) $w(0)=0$ and $w(1)=w(1+0)$ are assumed even if the point $x=0$ or $x=1$ beongs to the set $B_1$.

so that the relation (112) may be written in the form

$$T_m = 1 - \int_0^1 \frac{d}{dx}(x^m) \,.\, \Psi(x)\, dx - \int_0^1 \frac{d}{dx}(x^m) \,.\, W(x)\, dx. \qquad (115)$$

Now dividing the whole interval (0, 1) into $k$ sub-intervals $\delta_1, \delta_2, \cdots\cdots,$ $\delta_k$ of which $\delta_r$ has the end points $X_{r-1}$ and $X_r$ both belonging to $B_0$ at least except $X_0 = 0$, $X_k = 1$ we have, if $X_{r-1} < X_{rt} < X_r$,

$$X_{r-1}^m \{w(X_r) - w(X_{r-1})\} \leqq \sum_{t=1}^{\infty} X_{rt}^m \lambda_{rt}$$
$$\leqq X_r^m \{w(X_r) - w(X_{r-1})\}$$

where $\sum_{t=1}^{\infty} \lambda_{rt}$ denotes the sum or the limiting sum of the saltuses of the function $\bar{\lambda}(x)$ in the interval $(X_{r-1}, X_r)$ so that

$$w(X_r) - w(X_{r-1}) = \sum_{t=1}^{\infty} \lambda_{rt},$$

and therefore if $\lambda_1, \lambda_2, \lambda_3, \cdots\cdots$ denote the limiting probabilities $\pi(x-0, x+0)$ at points $x_1, x_2, x_3, \cdots\cdots$ of the set $B_1$ arranged in some order, then

$$\sum_{r=1}^{k} X_{r-1}^m \{w(X_r) - w(X_{r-1})\} \leqq \sum_{i=1}^{\infty} x_i^m \lambda_i$$
$$\leqq \sum_{r=1}^{k} X_r^m \{w(X_r) - w(X_{r-1})\},$$

i.e.

$$X_{k-1}^m w(1) - \sum_{r=1}^{k-1} \{X_r^m - X_{r-1}^m\} w(X_r) \leqq \sum_{i=1}^{\infty} x_i^m \lambda_i,$$

$$w(1) - \sum_{r=2}^{k} \{X_r^m - X_{r-1}^m\} w(X_{r-1}) \geqq \sum_{i=1}^{\infty} x_i^m \lambda_i.$$

These relations give rise to

$$w(1) - \int_0^1 \frac{d}{dx}(x^m)\, W(x)\, dx = \sum_{i=1}^{\infty} x_i^m \lambda_i, \qquad (116)$$

and hence, by aid of

$$w(1) = 1 - \Psi(1), \qquad (117)$$

we get

$$1 - \int_0^1 \frac{d}{dx}(x^m)\, W(x)\, dx = \sum_{i=1}^{\infty} x_i^m \lambda + \Psi(1).$$

The relation (115) then changes into

$$T_m = \sum_{i=1}^{\infty} x_i^m \lambda_i + \Psi(1) - \int_0^1 \frac{d}{dx}(x^m)\, \Psi(x)\, dx, \tag{118}$$

$$m = 1, 2, 3, \cdots\cdots.$$

Thus it has been shown that if all $T_m$ exist then the relations (118) must be satisfied for a continuous function $\Psi(x)$ and for a set of pairs of values of $(x_i, \lambda_i)$ $(i=1, 2, 3, \cdots\cdots)$ with the conditions

$$\left.\begin{array}{lll} 0 \leqq x_i \leqq 1, & x_i \neq x_k, & (i \neq k) \\ \lambda_i \geqq 0 (^1). & & \end{array}\right\} \tag{119}$$

Observe that the relation (117) is equivalent to the relation (118) for $m=0$.

Conversely if there exist a continuous function $\Psi(x)$ in the interval $(0, 1)$ and a set of pairs of values of $(x_i, \lambda_i)$ $(i=1, 2, 3, \cdots\cdots)$ subject to the conditions (119) so as to satisfy the system of relations

$$T_m = \sum_{i=1}^{\infty} x_i^m \lambda_i + \Psi(1) - \int_0^1 \frac{d}{dx}(x^m)\, \Psi(x)\, dx, \tag{118$'$}$$

$$m = 0, 1, 2, 3, \cdots\cdots,$$

then it can be shown without difficulty that, at points of an everywhere dense set complementary to the set of points $(x_i)$,

$$\pi(0, x) = \Psi(x) + \Sigma_x \lambda_i, \tag{120}$$

where $\Sigma_x$ denotes the summation about $\lambda_i$ corresponding to $x_i$ which lie in the interval $(0, x)$.

We have therefore;

*If $T_m$ $(m=1, 2, 3, \cdots\cdots)$ exist, then the system of equations (118)$'$ is always satisfied by a continuous function $\Psi(x)$ and a set of values of $(x_i, \lambda_i)$ subject to the conditions (119). The set of points $x_i$ corresponding to non-vanishing $\lambda_i$, in this case, forms the set $B_1$.*

In the case when $D\Psi(x)$ (2) which is any one of the four derivatives of $\Psi(x)$, is limited in the interval $(0, 1)$ such that $D\Psi(x)$ is integrable, or has at most points of infinite discontinuity forming a closed enumerable set such that $D\Psi(x)$ has a convergent improper integral, the equations (118)$'$ may be transformed into

$$T_m = \sum_{i=1}^{\infty} x_i^m \lambda_i + \int_0^1 x^m \lambda(x)\, dx, \tag{121}$$

---

(1) In case when there exists none or a finite number of points of the set $B_1$, all or all but a finite number of $\lambda_i$ are to vanish.

(2) It is evident that $D\Psi(x) \geqq 0$.

$$m=0, 1, 2, 3, \cdots\cdots$$

where

$$\lambda(x)=D\,\Psi(x)\,.$$

We have therefore:

*If there exist a non-negative function $\lambda(x)$ and a set of values of $(x_i, \lambda_i)$ with the conditions (119) satisfying a system of relation (121) then*

$$\pi(0, x)=\Sigma_x\,\lambda_i+\int_0^x \lambda(x)\,dx\,, \tag{122}$$

*at points of the set complementary to the set of points $x_i$, and the set of points $x_i$ corresponding to non-vanishing $\lambda_i$ is identical with the set $B_1$.*

**19.** We shall now go on to the classification of determinate systems. It has been shown that in a determinate system of trials the function $\pi(0, x)$, at points $x$ belonging to the set $B_0$, may be expressed in the form

$$\pi(0, x)=\psi(x)+w(x)\,,$$

where $\psi(x)$ is such a function of $x$ that its extended function $\Psi(x)$ is continuous in the whole interval $(0, 1)$ and $w(x)$ denotes the sum of the limiting probabilities $\pi(x-0, x+0)$ at points of the set $B_1$ contained in the interval $(0, x)$; and the condition to be satisfied by these functions has been found in two forms, one in (112) and another in (118).' It will be seen from the latter form that the case when $\Psi(x)$ vanishes identically throughout the whole interval $(0, 1)$ is characterized by the condition

$$T_m=\sum_{i=1}^{\infty} x_i^m\,\lambda_i\,, \tag{123}$$

$$m=0, 1, 2, 3, \cdots\cdots$$

Let us call such a system of trials the *simple determinate system.* It is evident that the set $A_0'$ defined in § 10 coincides with the set $A_0$ defined in § 1, in this case([1]).

Again if $B_0'$ denotes the complementary set of $A_0'$ in the set $B_0$ and $G$ be the sum set of $B_0'$ and $B_1$ then $G$ has been shown to be a closed set [§ 10] and therefore it may be resolved as a sum of an enumerable set $H$

([1]) It does not follow that if $A_0'$ coincides with $A_0$ in a determinate system, the system is simple. A simple example will be given in § 22 in which the set $A_0'$ does not exist and still the system of trials is not a simple determinate one.

and a perfect set $H_1$ of which the former can be expressed in the form

$$H = G_0 + G_1 + \cdots\cdots + G_\mu,$$

where $\mu$ is a finite ordinal number or a transfinite ordinal number of the second class and $G_i$ denotes the isolated set of points contained in the $i^{\text{th}}$ derivative of $G$. Let us call the point belonging to $G_i$ the point of the $(i+1)^{\text{th}}$ class and the system of trials for which every point in the set $G$ is at most of the $(i+1)^{\text{th}}$ class the *system of trials of the* $(i+1)^{\text{th}}$ *class* or *of the generalized* $(i+1)^{\text{th}}$ *class* according as $\Psi(x)$ vanishes or not identically. It is clear that for the occurrence of such a system of trials it is necessary and sufficient that the $i^{\text{th}}$ derivative of the set of points $(x_j)$ where $x_1, x_2, \cdots\cdots$ satisfy the system of relations (123) or (118)′ consists of a finite number of points[1].

Further in the case when the set $G$ contains a perfect set $H_1$ we can not classify the system of trials by the notion of class above mentioned. We distinguish such a case by denominating the *system of trials of the second kind.*

The set $B_1$, in this case, may be nowhere dense in the whole interval $(0, 1)$ or it may be every-where dense in some or whole parts of the interval in which case the set $G$ contains all points lying in some sub-intervals of $(0, 1)$. In the former case the system of trials is said to have *nowhere dense set of points of condensation of the relative frequency* and in the latter to have *every-where dense set of points of its condensation in some sub-intervals* of $(0, 1)$.

Evidently the conditions will at once be written down for the occurrence of the system of trials of the second kind having either nowhere dense set of points of condensation of the relative frequency or every-where dense sets of points of its condensation in some sub-intervals.

**20.** In the Part I it has been shown that if a system of trials be of the first class the analytic function $\phi(z)$ defined by

$$\phi(z) = \sum_{m=0}^{\infty} T_m z^m \tag{124}$$

represents a rational function of $z$ of a special character. The corresponding property of the function $\phi(z)$ for a simple determinate system of trials will now be considered.

---

(1) Here of course $x_j$ is assumed to correspond to the non-vanishing $\lambda_j$. In case when the set $H$ does not exist, the system of trials belongs to the zero class (generalized), answering the case when $i = -1$.

By a almost similar method to that adopted by Poincaré in the proof of the existence of a special lacunary function([1]), we may show that if the series $\Sigma \lambda_i$ be absolutely convergent and $x_i$ be a point of the set $B_1$, then

$$\sum_{i=0}^{\infty} \frac{\lambda_i}{1-x_i z}$$

represents a uniform analytic function of $z$, the set of singular points of which is identical with the set $\overline{B_1}$ of points $\left(\frac{1}{x_i}\right)$ closed by the addition of its limiting points, i.e. the set $\overline{G}$ which consists of reciprocal points of $G$([2]).

Since, if the system of trials belongs to a simple determinate system then the system of relations (123) holds, we have

$$\sum_{i=0}^{\infty} \frac{\lambda_i}{1-x_i z} = \sum_{i=0}^{\infty} \lambda_i \left\{ \sum_{m=0}^{\infty} (x_i z)^m \right\} = \sum_{m=0}^{\infty} \left\{ \sum_{i=0}^{\infty} x_i^m \lambda_i \right\} z^m$$

$$= \sum_{m=0}^{\infty} T_m z^m,$$

in the domain bounded by a circle of radius $r(<1)$ and with the origin as center, and therefore we obtain the result:

*If a system of trials belong to a simple determinate system and if $\overline{B}$ be the set consisting of points reciprocal to those of the set $B_1$ (excluding the point $x=\infty$ in case when $B_1$ contains the point $x=0$) then the function $\phi(z)$ defined by (124) is a uniform analytic function of $z$ having only singularities at points of the set $\overline{B}_1$ or of its derived set.*

Specially when the set $G$ which is the common measure of the set $B_1$ and its derivative is enumerable, that is, the system of trials belongs to the $(\mu+1)^{\text{th}}$ class where $\mu$ is some number of the first or second class, the uniform function $\phi(z)$ will be of the $(\mu+1)^{\text{th}}$ class, thus the class of the function $\phi(z)$ coinciding with that of the system of trials.

Again when the set $B_1$ is every-where dense in some sub-interval, say $(a, b)$, of $(0, 1)$ the function $\phi(z)$ will have the segment $\left(\frac{1}{b}, \frac{1}{a}\right)$ on the real axis as a line of singularities.

**21.** As an illustration of the preceding result we shall give here two examples belonging to the class considered.

---

([1]) Forsyth, Theory of Functions of a Complex Variable, 2. ed. (1900), p. 157.

([2]) In case when the point 0 belongs to the set $B_1$ but not to its derived set, the point $x=\infty$ must be excluded from the set $\overline{G}$.

(i) *A man is allowed to throw first with a die with which the probability of throwing ace is $x_1$; if he throws ace he is to continue to throw an unlimited number of times with that die; but if he throws another mark he is to throw for the second time with a second die with which the probability of throwing ace is $x_2$. If he throws ace this time he is to continue to throw an unlimited number of times with that die; but if he does not throw ace, he is to throw for the third time with a third die with which the probability of throwing ace is $x_3$ and so on indefinitely*(1). *Then if*

$$x_i \neq x_k \quad (i \neq k)(^2), \quad 1 > x_i > 0, \tag{125}$$

*the system of trials will belong to a simple determinate system with respect to the throwing of ace.*

If $p_m$ denote the probability of throwing ace in $m^{\text{th}}$ trial *à priori*, then we have easily

$$p_m = x_1^2 + (1-x_1)\,x_2^2 + \cdots\cdots + (1-x_1)(1-x_2)\cdots\cdots(1-x_{m-2})\,x_{m-1}^2 + (1-x_1)(1-x_2)\cdots\cdots(1-x_{m-1})x_m, \qquad m \geqq 2, \tag{126}$$

and in general if

$$W_{m_1, m_2, \cdots} \quad (m_1 < m_2 < \cdots\cdots)$$

denote the probability of throwing ace in $m_1^{\text{th}}$, $m_2^{\text{th}}$, $\cdots\cdots$ trials simultaneously, the occurrence or non-occurrence of ace in other trials being perfectly disregarded, then

$$W_{m_1, m_2, \cdots m_r} = x_1^{r+1} + (1-x_1)x_2^{r+1} + \cdots + (1-x_1)(1-x_2)\cdots(1-x_{m_1-2})x_{m_1-1}^{r+1} + (1-x_1)(1-x_2)\cdots\cdots(1-x_{m_1-1})x_{m_1}^r, \tag{127}$$

Now from the identity

$$\sum_{i=1}^{m} (1-x_1)\,1-x_2)\cdots(1-x_{i-1})x_i = 1-(1-x_1)(1-x_2)\cdots(1-x_m)(^3), \tag{128}$$

we obtain the convergence of the positive series

$$\sum_{i=1}^{\infty} (1-x_1)(1-x_2)\cdots\cdots(1-x_{i-1})\,x_i^r, \qquad r \geqq 1 \tag{129}$$

and thence also the vanishing of the last term $(1-x_1)(1-x_2)\cdots\cdots(1-x_{m_1-1})\,x_{m_1}^r$ of the right hand side expression in (127), when $m_1 = \infty$.

---

(1) It is evident that $0 \leqq x_i \leqq 1$ and by the assumption (125) none of $x_i$ may become zero or 1.

(2) This assumption is not, however, essentially a necessary one for our result.

(3) The term corresponding to $i=1$ in the summation on the left hand side is to be replaced by $x_1$, a similar remark being to be applied to the summation in (129).

Let the sum of the series (129) be denoted by $S_r$. Then by (127) we may take an integer $k$, corresponding to an arbitrarily chosen positive number $\varepsilon$, so as to satisfy

$$| W_{m_1, m_2, \cdots m_r} - S_{r+1} | < \varepsilon, \quad \text{for} \quad m_1 \geqq k \tag{130}$$

independent of $m_2, m_3, \cdots\cdots m_r$. Since for this fixed number $k$, the sum

$$\sum_{m_1 \leqq k-1} W_{m_1, m_2, \cdots m_r}$$

where $\sum\limits_{m_1 \leqq k-1}$ denotes the summation about $m_1, m_2, \cdots\cdots, m_r$ satisfying

$$1 \leqq m_1 \leqq k-1, \quad m_1 < m_2 < \cdots\cdots < m_r \leqq n,$$

is at most of the $(r-1)^{\text{th}}$ order with respect to $n$ when $n$ becomes infinite, we have

$$\overline{\lim_{n=\infty}} \frac{\Sigma W_{m_1, m_2, \cdots m_r}}{n^r} = \overline{\lim_{n=\infty}} \frac{\sum\limits_{m_1 \geqq k} W_{m_1, m_2, \cdots m_r}}{n^r},$$

where $\Sigma$ on the left and $\sum\limits_{m_1 \geqq k}$ on the right denote the sums with respect to $m_1, m_2, \cdots\cdots, m_r$ satisfying

$$1 \leqq m_1 < m_2 < \cdots\cdots < m_r \leqq n,$$

and

$$k \leqq m_1 < m_2 < \cdots\cdots < m_r \leqq n,$$

respectively and therefore by (130) we get

$$\left| r! \ \overline{\lim_{n=\infty}} \frac{\Sigma W_{m_1, m_2, \cdots\cdots m_r}}{n^r} - S_{r+1} \right| < \varepsilon.$$

This relation can only be satisfied when

$$r! \lim_{n=\infty} \frac{\Sigma W_{m_1, m_2, \cdots\cdots m_r}}{n^r} = S_{r+1},$$

and consequently by the formula (37) [Part I § 7] we shall obtain

$$\begin{aligned} T_r &= S_{r+1} \\ &= \sum_{i=1}^{\infty} (1-x_1)(1-x_2)\cdots\cdots(1-x_{i-1})\, x_i^{r+1}. \end{aligned} \tag{131}$$

$$r = 1, 2, 3, \cdots\cdots\cdots\cdots$$

We now distinguish two cases separately.

(a) *Case when*

$$\prod_{i=1}^{\infty}(1-x_i)=0.$$

In this case the relation (128) gives

$$1=\sum_{i=1}^{\infty}(1-x_1)(1-x_2)\cdots\cdots(1-x_{i-1})x_i, \tag{132}$$

which, combined with (131), shows us that the system of relations

$$T_m=\sum_{i=1}^{\infty}X_i^m\lambda_i \tag{133}$$

$$m=0, 1, 2, \cdots\cdots\cdots\cdots$$

is satisfied by

$$X_i=x_i \quad \text{and} \quad \lambda_i=(1-x_1)(1-x_2)\cdots\cdots(1-x_{i-1})x_i.$$

$$i=1, 2, 3, \cdots\cdots\cdots\cdots$$

Thence we have the result:

*The system of trials belongs to a simple determinate system, the set* $(x_i)$ *coinciding with the set* $B_1$.

(b) *Case when*

$$\prod_{i=1}^{\infty}(1-x_i)\neq 0.$$

In this case putting

$$\prod_{i=1}^{\infty}(1-x_i)=l,$$

which necessarily exists, we get, in place of (132),

$$1=l+\sum_{i=1}^{\infty}(1-x_1)(1-x_2)\cdots\cdots(1-x_{i-1})x_i,$$

so that the system of relations (133) is satisfied by

$$X_1=0, \qquad \lambda_1=l,$$

$$X_{i+1}=x_i, \quad \lambda_{i+1}=(1-x_1)(1-x_2)\cdots\cdots(1-x_{i-1})x_i,$$

$$i=1, 2, 3, \cdots\cdots\cdots\cdots$$

and thus we have the result;

*The system of trials belongs to a simple determinate system, the set* $B_1$ *consisting of* $x=0$ *and points of the set* $(x_i)$.

If, in either one of the cases, the points $x_i$ $(i=1, 2, 3, \cdots\cdots)$ form an

every-where dense set in the interval $(0, 1)$—if for example the set $(x_i)$ coincides with the set of rational numbers contained in $(0, 1)$—, then we get a simple determinate system of the second kind having every-where dense set of points of condensation of the relative frequency in the whole interval $(0, 1)$. It has been shown, even in such a case, that the probability of the existence of

$$X_1 \leqq \frac{s}{n} \leqq X_2,$$

where $s$ denotes the number of the happening of ace in $n$ trials, and $X_1$, $X_2$ are two points of the set complementary to $B_1$, tends to a limit, when $n$ becomes infinite,

$$\Sigma' (1-x_1)(1-x_2) \cdots\cdots (1-x_{i-1})\, x_i,$$

where $\Sigma'$ denotes the summation about $i$ for which

$$X_1 < x_i < X_2$$

holds good.

(ii) *A man is allowed to throw first with a two-faced die marked* 1 *and* 2*; if he throws ace he is to throw, at* $m^{\text{th}}$ *trial, with a die of* $m+1$ *faces marked* $2, 1, 1, \cdots\cdots, 1$ *or* $1, 2, 2, \cdots, 2$ *according as he throws ace or deuce in the* $(m-1)^{\text{th}}$ *trial. On the contrary, if he throws deuce for the first time with the two-faced die he is to continue his throwing on the same conditions as in the example* (i) *beginning for the second time with a die with which the probability of throwing ace is* $x_1$. *Then if*

$$x_i \neq x_k, \quad (i \neq k)\ (^1), \quad 1 > x_i > 0, \tag{134}$$

*the system of trials will belong to a determinate system with respect to his throwing of ace.*

If $p_m$ denote the probability of throwing ace in $m^{\text{th}}$ trial *à priori*, then we have

$$p_m = \frac{1}{2} p_m' + \frac{1}{2} p_m'', \quad (m \geqq 2), \tag{135}$$

where $p_m'$ is the probability of throwing ace in $m^{\text{th}}$ trial *under the supposition that on the first trial the ace presents itself*, so that it may be found from the recurring formula

$$p_2' = \frac{2}{3},$$

(1) This assumption is not essentially a necessary one for our result.

$$p'_m = p'_{m-1}\frac{m}{m+1} + (1-p'_{m-1})\frac{1}{m+1}, \quad m=2,3,\cdots\cdots \quad (136)$$

and $p''_m$ is the probability of throwing ace in $m^{\text{th}}$ trial *under the supposition that on the first trial the deuce presents itself,* so that it will be given by the expression (126) for $m$ replaced by $m-1$, namely by

$$p''_m = \sum_{i=2}^{m-1}(1-x_1)(1-x_2)\cdots\cdots(1-x_{i-2})\,x_{i-1}^2$$
$$+(1-x_1)(1-x_2)\cdots\cdots(1-x_{m-2})\,x_{m-1}. \qquad (136)$$

Now the relation (136) gives, by aid of the formula (54), [Part I § 8]

$$\lim_{m=\infty} p_m' = \frac{1}{2}\ (^1),$$

and the relation (136)′ gives also

$$\lim_{m=\infty} p''_m = \sum_{i=1}^{\infty}(1-x_1)(1-x_2)\cdots\cdots(1-x_{i-1})\,x_i^2.$$

Thence we get

$$T_1 = \lim_{m=\infty} p_m = \frac{1}{2}\left\{\frac{1}{2} + \sum_{i=1}^{\infty}(1-x_1)(1-x_2)\cdots\cdots(1-x_{i-1})\,x_i^2\right\}. \quad (137)$$

Again in general if

$$W_{m_1, m_2, \cdots} \qquad (m_1 < m_2 < \cdots\cdots)$$

denote the probability of throwing ace in $m_1^{\text{th}}$, $m_2^{\text{th}}$, ……trials simultaneously, then

$$W_{m_1, m_2, \cdots, m_r} = \frac{1}{2}W'_{m_1, m_2, \cdots, m_r} + \frac{1}{2}W''_{m_1, m_2, \cdots, m_r},$$
$$(m_1 \geqq 2)$$

where $W'_{m_1, m_2, \cdots, m_r}$ and $W''_{m_1, m_2, \cdots, m_r}$ are the corresponding probabilities under the assumption that on the first trial (i) the ace presents itself and (ii) the deuce presents itself respectively.

Since we have by (85) [Part II § 13]

$$r!\lim_{n=\infty}\frac{1}{n^r}\sum_{m_1 \geqq 2} W'_{m_1, m_2, \cdots, m_r} = \frac{1}{r+1}\ (^2),$$

and by (131)

---

(1) This case corresponds to $P'=1$ and $Q=1$ in the formula (54).

(2) This can be obtained by finding $T_r$ from the formula (85) wherein $P'=1$, $Q=1$.

$$r!\lim_{n=\infty}\frac{1}{n^r}\sum_{m_1\geqq 2}W''_{m_1,m_2,\cdots,m_r}=\sum_{i=1}^{\infty}(1-x_1)(1-x_2)\cdots\cdots(1-x_{i-1})\,x_i^{r+1},$$

where $\sum\limits_{m_1\geqq 2}$ denotes the summation([1]) about $m_1, m_2,\cdots\cdots, m_r$ satisfying

$$2\leqq m_1<m_2<\cdots\cdots<m_r\leqq n\,,$$

it follows that

$$r!\lim_{n=\infty}\frac{\sum\limits_{m_1\geqq 2}W_{m_1,m_2,\cdots,m_r}}{n^r}=\frac{1}{2}\left\{\frac{1}{r+1}+\sum_{i=1}^{\infty}(1-x_1)\cdots\cdots(1-x_{i-1})\,x_i^{r+1}\right\},$$

and therefore

$$T_r=\frac{1}{2}\left\{\frac{1}{r+1}+\sum_{i=1}^{\infty}(1-x_1)(1-x_2)\cdots\cdots(1-x_{i-1})\,x_i^{r+1}\right\},\quad (138)$$

$$r=2,3,4,\cdots\cdots.$$

The two relations (137) and (138) show us that all $T_r\,(r=1,2,3,\cdots\cdots)$ exist and hence by the result in § 17 we see that the system of trials belongs to a determinate system with respect to the throwing of ace.

Again from the relations (137) and (138) it results that the system of relations

$$T_m=\sum_{i=1}^{\infty}X_i^m\,\lambda_i+\int_0^1 x^m\,\lambda(x)\,dx\,,\quad m=0,1,2,\cdots\cdots$$

is satisfied by

$$X_i=x_i,\quad \lambda_i=\frac{1}{2}(1-x_1)\cdots\cdots(1-x_{i-1})\,x_i\quad (i=1,2,3,\cdots\cdots)$$

$$\lambda(x)=\frac{1}{2}\,,$$

in case when $\prod\limits_{i=1}^{\infty}(1-x_i)$ vanishes and by

$$X_1=0,\quad \lambda_1=\frac{l}{2}\,,$$

$$X_{i+1}=x_i,\quad \lambda_{i+1}=\frac{1}{2}(1-x_1)\cdots\cdots(1-x_{i-1})\,x_i,\quad (i=1,2,3,\cdots\cdots)$$

---

([1]) It is evident that the two limits $\lim\limits_{n=\infty}\frac{1}{n^r}\Sigma W_{m_1 m_2\cdots m_r}$ and $\lim\limits_{n=\infty}\sum\limits_{m_1\geqq 2}W_{m_1 m_2\cdots m_r}$ are the same; for the difference $\Sigma W_{m_1 m_2\cdots m_r}-\sum\limits_{m_1\geqq 2}W_{m_1 m_2\cdots m_r}$ is at most of $(r-1)^{\text{th}}$ order with respect to $n$ when $n$ becomes infinite.

$$\lambda(x)=\frac{1}{2},$$

in case when $\prod_{i=1}^{\infty}(1-x_i)=l$ does not vanish. Consequently we have:

*The system of trials belongs to a determinate system, but not simple, the set $B_1$ consisting of, in the former case, points of the set $(x_i)$ and, in the latter, $x=0$ together with points of the set $(x_i)$.*

If, in this example, the points $x_i$ $(i=1, 2, 3, \cdots\cdots)$ form an everywhere dense set in the interval $(0, 1)$ then we get a determinate system (but not simple) of the second kind having everywhere dense set of points of condensation of the relative frequency in the whole interval $(0,1)$. It has been shown, even in such a case, that the probability of the existence of

$$X_1 \leqq \frac{s}{n} \leqq X_2,$$

where $s$ denotes the number of the happening of ace in $n$ trials, and $X_1$, $X_2$ are two points of the set complementary to $B_1$, tends to a limit

$$\frac{1}{2}\{(X_2-X_1)+\Sigma(1-x_1)(1-x_2)\cdots\cdots(1-x_{i-1})\,x_i\}\ (^1),$$

when $n=\infty$, where the symbol $\Sigma$ denotes the summation about $i$ for which $X_1<x_i<X_2$ holds good.

**22.** In the Part II we have given an expression of

$$\bar{\lambda}(x)=\pi(0, x),$$

in case when the system of trials belongs to a generalized zero class, starting from the system of integral equations (91) which $\bar{\lambda}(x)$ satisfies. From the analogy of two relations (91) and (112) it will be seen that, in case when the system of trials belongs to a determinate system, $\bar{\lambda}(x)$ is expressible in the form

$$\bar{\lambda}(x)=\frac{1}{\pi}\int_0^{\infty}\left[\left\{\sum_{m=0}^{\infty}(-1)^m\frac{T'_{2m+1}}{(2m+1)!}\,y^{2m}\right\}\cos xy\right.$$
$$\left.+\left\{\sum_{m=0}^{\infty}(-1)^{m+1}\frac{T'_{2m}}{(2m)!}\,y^{2m-1}\right\}\sin xy\right]dy\ (^2), \qquad (139)$$

---

(1) The continuous function $\Psi(x)$ defined in § 18 becomes $\frac{x}{2}$ in this case and therefore we have

$$\Psi(X_2)-\Psi(X_1)=\frac{1}{2}(X_2-X_1).$$

(2) This result can also be obtained directly without using Fourier's theorem.

at a point $x$ of the set $B_0$ in the open interval $(0, 1)$ while at a point $x$ of the set $B_1$, $\frac{1}{2}\{\bar{\lambda}(x+0)+\bar{\lambda}(x-0)\}$ is represented by the same expression. This is a direct consequence from Fourier's theorem.

Then it follows that, within the open interval $(0, 1)$, the function represented by the integral standing on the right of (139) is monotonic, continuous and coincident with $\pi(0, x)$ at points of the set $B_0$ while it is certainly discontinuous and has the same saltus(1) as the function $\pi(0, x)$ at points of the set $B_1$, and hence we have:

*If all $T_m$ exist, then the integral standing on the right of (139) certainly exists and is a monotonic function of $x$ within the open interval $(0, 1)$. Within the open interval $(0, 1)$ the set of points of its discontinuity coincides with the set $B_1$ the saltus of the function at any one of these points being equal to the limiting probability $\pi(x-0, x+0)$ at that point; at any point of its continuity it represents the limiting probability $\pi(0, x)$.*

The point $x=0$ does or does not belong to the set $B_0$ according as the integral for $x=0$ vanishes or not, and the point $x=1$ does or does not belong to the set $B_0$ according as the integral for $x=1$ has the value $\frac{1}{2}$ or not(2).

---

(1) This function is not necessarily identical with $\pi(0, x)$ at points of the set $B_1$.

(2) The formula (139) may also be written in the form, if $0<x<1$,

$$\bar{\lambda}(x)=\frac{1}{2}-\frac{1}{\pi}\int_0^\infty\left[\left\{\sum_{m=0}^{\infty}(-1)^m\frac{T_{2m+1}}{(2n+1)!}y^{2m}\right\}\cos xy\right.$$

$$\left.+\left\{\sum_{m=0}^{\infty}(-1)^{m+1}\frac{T_{2m}}{(2m)!}y^{2m-1}\right\}\sin xy\right]dy.$$

This formula might serve us for the calculation of $\bar{\lambda}(x)$. For example if

$$T_m=\frac{1}{m+1}\quad(m=0, 1, 2, \cdots),$$

then

$$\left\{\sum_{m=0}^{\infty}(-1)^m\frac{T_{2m+1}}{(2m+1)!}y^{2m}\right\}\cos xy+\left\{\sum_{m=0}^{\infty}(-1)^{m+1}\frac{T_{2m}}{(2m)!}y^{2m-1}\right\}\sin xy$$

$$=\frac{1}{y^2}(1-\cos y)\cos xy-\frac{1}{y^2}\sin y\sin xy$$

$$=\frac{\cos xy-\cos(1-x)y}{y^2},$$

and therefore

## Part IV.

## System of Trials of the First Class with respect to a System of Mutually Exclusive Events.

**23.** In the foregoing three Parts we have restricted our investigation only to the case where an event is concerned with. We shall now go on to derive analogous results for a system of mutually exclusive events. We first take the case which corresponds to that considered at Part I.

Let $s_1, s_2, \cdots\cdots, s_l$ denote the numbers of the happening of $l$ mutually exclusive events in $n$ trials respectively such that

$$s_1+s_2+\cdots\cdots+s_l \leqq n\,.$$

A point $t=(t_1, t_2, \cdots\cdots, t_l)$ within the closed domain $D$(1) bounded by the coordinate planes

$$t_i=0, \quad (i=1, 2, \cdots\cdots, l)$$

and a plane $t_1+t_2+\cdots\cdots+t_l=1$ in $l$-dimensional space is called the *point of the class* 0 if, by taking a sufficiently small positive number $\varepsilon$, the probability $\pi_n(t\,;\varepsilon)$ that $s_1, s_2, \cdots\cdots, s_l$ will satisfy simultaneously

$$\left|\frac{s_1}{n}-t_1\right| \leqq \varepsilon, \quad \left|\frac{s_2}{n}-t_2\right| \leqq \varepsilon, \cdots\cdots, \quad \left|\frac{s_l}{n}-t_l\right| \leqq \varepsilon \text{ (2)},$$

approaches zero as limit when $n$ becomes infinite and the set of such points is denoted by $A_0$(3). Similarly a point $t$ within the domain

---

$$\overline{\lambda}(x)=\frac{1}{2}-\frac{1}{\pi}\int_0^\infty \frac{\cos xy-\cos(1-x)y}{y^2}dy$$
$$=x, \qquad \text{if } 0<x<1\,.$$

But in many instances it seems to require a laborious calculation to evaluate $\pi(0, x)$ by means of this formula.

(1) This means "in the interior or on the boundary of"

(2) It is to be taken into account that the probability that the point $\left(\frac{s_1}{n}, \frac{s_2}{n}, \cdots\cdots, \frac{s_l}{n}\right)$ lies outside the domain $D$ always vanishes.

(3) If we extend this definition to external points of the domain $D$, it is evident that all such points belong to this set $A_0$. For this reason we shall frequently regard points external to $D$ as belonging to the set $A_0$.

indicated is called the *point of the first class,* if the probability $\pi_n(t;\varepsilon)$ that $s_1, s_2, \cdots\cdots, s_l$ will satisfy simultaneously

$$\left|\frac{s_1}{n}-t_1\right| \leqq \varepsilon, \quad \left|\frac{s_2}{n}-t_2\right| \leqq \varepsilon, \cdots\cdots \quad \left|\frac{s_l}{n}-t_l\right| \leqq \varepsilon,$$

approaches a definite positive number $\lambda$ *for all positive* $\varepsilon$ smaller than a fixed positive number $\varepsilon_1$, and the set of such points is denoted by $A_1$.

We have then a theorem:

*If all points within a closed domain $D'$ which is a part of $D$ belong to the set $A_0$ then the probability $\pi_n(D')$ that the point $\left(\frac{s_1}{n}, \frac{s_2}{n}, \cdots\cdots, \frac{s_l}{n}\right)$ will lie in the interior or on the boundary of $D'$ tends to zero as limit when $n$ becomes infinite.*

For if every point in the domain $D'$ in $l$-dimensional continuum belong to the set $A_0$ then corresponding to any point in the domain an area enclosing that point can always be found such that the probability that the point $\left(\frac{s_1}{n}, \frac{s_2}{n}, \cdots, \frac{s_l}{n}\right)$ will lie in the interior or on the boundary of the area tends to zero as limit when $n=\infty$, so that by Heine-Borel's theorem(1) we can choose, from the set of overlapping areas so formed, a finite number of areas, to one at least of which every point in the domain $D'$ is interior. Since the finite sum of the separate probabilities that the point $\left(\frac{s_1}{n}, \frac{s_2}{n}, \cdots\cdots, \frac{s_l}{n}\right)$ will lie in the interior or on the boundary of those areas has clearly the limit 0 when $n$ becomes infinite and still is not smaller than $\pi_n(D')$ we shall have

$$\lim_{n=\infty} \pi_n(D')=0,$$

which is to be proved.

From this we may infer that *the set $A_0$ can not consist of all points of the continuum $D$.*

Furthermore just as in § 1 it is easy to show that

*The set $C(A_0)$, complementary to $A_0$ in the domain $D$, forms a closed set.*

*In case when there exists no point in the domain $D$ which belongs neither to $A_0$ nor to $A_1$ the set $A_1$ must consist of at least one point and at most a finite number of points.*

---

(1) Hobson, l. c., p. 88.

Such a system of trials is said to be of the *first class* with respect to the system of mutually exclusive events under consideration.

**24.** We shall now find conditions necessary and sufficient for the existence of the two sets $A_0$ and $A_1$ only, i. e. of the system of trials of the first class.

To simplify the calculation, let us consider the case when $l=2$, the reasoning, however, being applicable in the general case.

If all points in the two-dimensional domain $D$ bounded by the coordinate axes and the line $x+y=1$ belong to either the set $A_0$, or the set $A_1$, then the latter set $A_1$ has been shown to be a finite set. Let $X_1, X_2, \cdots\cdots, X_r$ be the points of the set $A_1$, the different abscissae and the different ordinates of these points being $x_1, x_2, \cdots\cdots, x_{r_1}$ and $y_1, y_2, \cdots\cdots, y_{r_2}$ respectively. Draw lines through the points parallel to the coordinate axes thus forming $r_1 r_2$ points of intersection whose coordinates are

$$x=x_i\,, \qquad y=y_j\,,$$

$$i=1, 2, \cdots\cdots, r_1\,; \quad j=1, 2, \cdots\cdots, r_2\,,$$

and let the set of such points of intersection be denoted by $I$ (1). Further let $I_x$ and $I_y$ be two linear sets of points $(x_i)$ and $(y_j)$ which are the projections of the points of $I$ on the $x$- and the $y$- axes respectively.

If we denote by $F(n\,;\,s_1, s_2)$ the probability of the happening $s_1$ and $s_2$ times respectively of two mutually exclusive events in $n$ trials such that

$$s_1+s_2 \leqq n\,,$$

and consider the expression

$$T_{m_1, m_2}(n)=\sum_{0\leqq s_1+s_2\leqq n}\left(\frac{s_1}{n}\right)^{m_1}\left(\frac{s_2}{n}\right)^{m_2}F(n\,;\,s_1, s_2)\,. \quad (140)$$

$$(m_1 \geqq 0,\ m_2 \geqq 0)$$

then we have easily

$$\lim_{n=\infty} T_{m_1, m_2}(n)=\sum_{i=1}^{r_1}\sum_{j=1}^{r_2} x_i^{m_1}\, y_j^{m_2}\cdot\lambda_{ij}\,,$$

(1) Points of I which lie outside the domain $D$, if such exist, will conveniently be looked upon as belonging to the set $A_0$, as we have remarked before.

where $\lambda_{ij}$ ([1]) is the limiting probability $\pi(t;\varepsilon)$ at the point $(x_i, y_j)$ so that by putting

$$T_{m_1, m_2} = \lim_{n=\infty} T_{m_1, m_2}(n),$$

we obtain the existence of $T_{m_1, m_2}$.

Moreover since the set $I_x$ is identical with the linear set of points $x$ for which the probability of the existence of

$$\left| \frac{s_1}{n} - x \right| \leqq \varepsilon,$$

approaches a definite positive number, the system of equations

$$T_{m_1, 0} = \sum_{i=1}^{r_1} x_i^{m_1} \lambda_i, \tag{141}$$

$$m_1 = 0, 1, 2, \cdots\cdots, 2r_1,$$

will have a unique solution for the case under consideration([2]). Similarly we obtain a unique solution of

$$T_{0, m_2} = \sum_{j=1}^{r_2} y_j^{m_2} \lambda_j', \tag{142}$$

$$m_2 = 0, 1, 2, \cdots\cdots, 2r_2.$$

We have therefore:

In order that a system of trials may be of the first class it is necessary that the two systems of equations (141) and (142) should have unique solutions for some integers $r_1$ and $r_2$ respectively and also

$$T_{m_1, m_2} \ (m_1 \geqq 0, \ m_2 \geqq 0)$$

should exist.

We shall now show conversely that if there exist a unique solution of (141) as well as a unique solution of (142) and also

$$T_{m_1, m_2} \begin{pmatrix} m_1 = 1, 2, \cdots\cdots, r_1 - 1 \\ m_2 = 1, 2, \cdots\cdots, r_2 - 1 \end{pmatrix}$$

exists, then every point in the domain $D$ belongs either to the set $A_0$ or to the set $A_1$ so that the system of trials belongs to the first class.

---

([1]) $\lambda_{ij}$, in this case, may or may not vanish according as the point $(x_i, y_j)$ belongs to the set $A_0$ or to the set $A_1$.

([2]) By the result in § 5.

It is evident from the existence of a unique solution of (141)(1) that if $x$ be any point not coinciding with any one of the points $x_1, x_2, \cdots\cdots, x_{r_1}$ which satisfy (141) the probability of the existence of

$$\left|\frac{s_1}{n}-x\right| \leqq \varepsilon,$$

approaches zero as limit when $n=\infty$, and also from the existence of a unique solution of (142) that if $y$ be any point not coincident with any one of the points $y_1, y_2, \cdots\cdots, y_{r_2}$ which satisfy (142) the probability of the existence of

$$\left|\frac{s_2}{n}-y\right| \leqq \varepsilon,$$

approaches zero as limit when $n=\infty$. It follows therefore that any point in the domain $D$ which does not coincide with any one of the points of intersection of two systems of parallel lines

$$x=x_i, \quad (i=1, 2, \cdots\cdots, r_1)$$

and

$$y=y_j, \quad (j=1, 2, \cdots\cdots, r_2)$$

belongs to the set $A_0$, and hence it will be sufficient for our present purpose to show that any one of the points $(x_i, y_j)$ where $x_i$ and $y_j$ are such as to satisfy (141) and (142) respectively, belongs to either the set $A_0$ or the set $A_1$.

Now if the simultaneous equations (141) have a unique solution, then since no two of $x_1, x_2, \cdots\cdots, x_{r_1}$ are equal we have, for those values of $x_1, x_2, \cdots\cdots, x_{r_1}$, a unique set of $\lambda_1(m_2), \lambda_2(m_2), \cdots\cdots, \lambda_{r_1}(m_2)$ satisfying

$$T_{m_1, m_2}=\sum_{i=1}^{r_1} x_i^{m_1} \lambda_i(m_2),$$

$$m_1=0, 1, 2, \cdots\cdots, r_1-1,$$

provided that $T_{m_1, m_2}$ exist.

Again if the simultaneous equations (142) have a unique solution then since no two of $y_1, y_2, \cdots\cdots, y_{r_2}$ are equal we have, for those values of $y_1, y_2, \cdots\cdots, y_{r_2}$, a unique set of values of $\lambda_{i1}, \lambda_{i2}, \cdots\cdots, \lambda_{ir_2}$ satisfying

$$\lambda_i(m_2)=\sum_{j=1}^{r_2} y_j^{m_2} \lambda_{ij},$$

(1) By the result in § 5.

$$m_2=0, 1, 2, \cdots\cdots, r_2-1 .$$

Consequently for the values of $x_1, x_2, \cdots\cdots, x_{r_1}$ and $y_1, y_2, \cdots\cdots, y_{r_2}$ satisfying (141) and (142) respectively we have a unique solution

$$\lambda_{ij} \qquad \begin{pmatrix} i=1, 2, \cdots\cdots, r_1 \\ j=1, 2, \cdots\cdots, r_2 \end{pmatrix}$$

of the system of linear equations

$$T_{m_1, m_2}=\sum_{i=1}^{r_1} x_i^{m_1} \left\{ \sum_{j=1}^{r_2} y_j^{m_2} \lambda_{ij} \right\}, \tag{143}$$

$$m_1=0, 1, 2, \cdots\cdots, r_1-1 ,$$

$$m_2=0, 1, 2, \cdots\cdots, r_2-1 ,$$

provided that

$$T_{m_1, m_2} \quad \begin{pmatrix} m_1=0, 1, 2, \cdots\cdots, r_1-1 \\ m_2=0, 1, 2, \cdots\cdots, r_2-1 \end{pmatrix}$$

exist.

If $\lambda_{ij}$ be so taken, it can be shown that the point $(x_i, y_j)$ is one at which the probability $\pi(t; \varepsilon)=\lambda_{ij}$ and hence it belongs to the set $A_0$ or to the set $A_1$ according as $\lambda_{ij}$ vanishes or not( [1] ).

To show this we consider the expression $E_{ij}(n)$ defined by

$$E_{ij}(n)=\sum_{0 \leqq s_1+s_2 \leqq n} f_{1i}\left(\frac{s_1}{n}\right) . f_{2j}\left(\frac{s_2}{n}\right) F(n; s_1, s_2),$$

where

$$f_{1i}(x)=\frac{A_1(x)}{x-x_i}, \qquad A_1(x)=(x-x_1)(x-x_2)\cdots\cdots(x-x_{r_1}),$$

$$f_{2j}(y)=\frac{A_2(y)}{y-y_j}, \qquad A_2(y)=(y-y_1)(y-y_2)\cdots\cdots(y-y_{r_2}).$$

Then since $f_{1i}(x)$ is a polynomial of the degree $r_1-1$ and $f_{2j}(y)$ of the degree $r_2-1$ we have

$$f_{1i}(x)\, f_{2j}(y)=\sum a_{m_1 m_2}\, x^{m_1}\, y^{m_2},$$

the summation being extended over

$$m_1=0, 1, 2, \cdots\cdots, r_1-1,$$

$$m_2=0, 1, 2, \cdots\cdots, r_2-1,$$

---

( [1] ) Some of $\lambda_{ij}$ may vanish but not all of $\lambda_{i1}, \lambda_{i2}, \cdots, \lambda_{ir_2}$ or $\lambda_{1j}, \lambda_{2j}, \cdots, \lambda_{jr_1}$.

and therefore

$$\lim_{n=\infty} E_{ij}(n) = \lim_{n=\infty} \sum_{0 \leqq s_1+s_2 \leqq n} \left\{ \sum_{m_1, m_2} a_{m_1 m_2} \left(\frac{s_1}{n}\right)^{m_1} \left(\frac{s_2}{n}\right)^{m_2} \right\} F(n; s_1, s_2)$$

$$= \sum_{m_1, m_2} a_{m_1, m_2} T_{m_1, m_2}, \tag{144}$$

so that by (143)

$$\lim_{n=\infty} E_{ij}(n) = \sum_{i'=1}^{r_1} \sum_{j'=1}^{r_2} \left\{ \sum_{m_1, m_2} a_{m_1 m_2} x_{i'}^{m_1} y_{j'}^{m_2} \right\} \lambda_{i'j'}$$

$$= \sum_{i'=1}^{r_1} \sum_{j'=1}^{r_2} f_{1i}(x_{i'}) f_{2j}(y_{j'}) \lambda_{i'j'}$$

$$= f_{1i}(x_i) \cdot f_{2j}(y_j) \cdot \lambda_{ij}. \tag{145}$$

On the other hand we get, just in the same manner as in § 3,

$$\lim_{n=\infty} E_{ij}(n) = \lim_{n=\infty} \Sigma_{ij} f_{1i}\left(\frac{s_1}{n}\right) f_{2j}\left(\frac{s_2}{n}\right) F(n; s_1, s_2),$$

where $\Sigma_{ij}$ denotes the summation about $s_1$ and $s_2$ satisfying

$$\left| \frac{s_1}{n} - x_i \right| \leqq \varepsilon, \qquad \left| \frac{s_2}{n} - y_j \right| \leqq \varepsilon,$$

$\varepsilon$ being a positive number which may be taken arbitrarily small.

This, combined with (145), leads us ultimately to

$$\lim_{n=\infty} \Sigma_{ij} F(n; s_1, s_2) = \lambda_{ij},$$

which is to be proved.

The results obtained in the present article may thus be expressed as follows:

*In order that a system of trials may be of the first class, with respect to a system of two mutually exclusive events, it is necessary and sufficient that there should exist two positive integers $r_1$ and $r_2$ such that the two systems of the simultaneous equations (141) and (142) will have unique solutions respectively and also*

$$T_{m_1, m_2} \quad \begin{pmatrix} m_1 = 0, 1, 2, \cdots\cdots, r_1 - 1 \\ m_2 = 0, 1, 2, \cdots\cdots, r_2 - 1 \end{pmatrix} \tag{146}$$

*should exist.*

*If this condition be satisfied we may find the limiting probability $\lambda_{ij}$ by solving the equations (143) for values of $x_1, x_2, \cdots\cdots, x_{r_1}$ and $y_1, y_2 \cdots\cdots, y_{r_2}$ which satisfy respectively (141) and (142); or more simply by*

*equating the right-hand side expressions of (144) and (145) namely by*

$$\lambda_{ij}=\frac{\Sigma\, a_{m_1 m_2}\, T_{m_1 m_2}}{f_{1i}(x_i)\, f_{2j}(y_j)}\,.$$

*The values of $\lambda_{ij}$ so found may vanish, in which case the point $(x_i, y_j)$ belongs to the set $A_0$; or may not vanish, in which case the point $(x_i, y_j)$ belongs to the set $A_1$, so that the number*(1) *of the non-vanishing $\lambda_{ij}$ gives that of the points belonging to the set $A_1$.*

**25.** The reasoning given in the last article may be applied to the general case.

Denoting by $F(n\,;s_1, s_2, \cdots\cdots, s_l)$ the probability of the happening, $s_1, s_2, \cdots\cdots, s_l$ times, of $l$ mutually exclusive events in $n$ trials such that

$$s_1+s_2+\cdots\cdots+s_l\leqq n,$$

and putting

$$T_{m_1 m_2\cdots\cdots m_l}(n)=\Sigma\left(\frac{s_1}{n}\right)^{m_1}\left(\frac{s_2}{n}\right)^{m_2}\cdots\cdots\left(\frac{s_l}{n}\right)^{m_l} F(n\,;s_1, s_2, \cdots\cdots, s_l),$$

the summation being extended with respect to $s_1, s_2, \cdots\cdots, s_l$ over

$$0\leqq s_1+s_2+\cdots\cdots+s_l\leqq n,$$

and

$$T_{m_1, m_2, \cdots\cdots, m_l}=\lim_{n=\infty} T_{m_1, m_2\cdots\cdots, m_l}(n)\,,$$

we have the result:

*In order that a system of trials may be of the first class with respect to a system of $l$ mutually exclusive events it is necessary and sufficient that there should exist $l$ positive integers $r_1, r_2, \cdots\cdots, r_l$ such that each of the systems of the simultaneous equations*

$$\left.\begin{array}{ll}
T_{m_1, 0, 0, \cdots, 0}=\sum\limits_{i=1}^{r_1} t_{1i}^{m_1}\lambda_i\,, & m_1=0, 1, 2, \cdots\cdots, 2r_1 \\
T_{0, m_2, 0, \cdots, 0}=\sum\limits_{i=1}^{r_2} t_{2i}^{m_2}\lambda_i'\,, & m_2=0, 1, 2, \cdots\cdots, 2r_2 \\
\cdots\cdots\cdots\cdots & \cdots\cdots\cdots\cdots \\
T_{0, 0, 0, \cdots, m_l}=\sum\limits_{i=1}^{r_l} t_{li}^{m_l}\lambda_i^{(l-1)}, & m_l=0, 1, 2, \cdots\cdots, 2r_l
\end{array}\right\}\quad (147)$$

(1) This number of course is not greater than $r_1 r_2$ and is not smaller than the greater of $r_1$ and $r_2$.

*will have unique solution and also*

$$T_{m_1, m_2, \cdots\cdots, m_l} \begin{pmatrix} m_1=0, 1, 2, \cdots\cdots, r_1-1 \\ m_2=0, 1, 2, \cdots\cdots, r_2-1 \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ m_l=0, 1, 2, \cdots\cdots, r_l-1 \end{pmatrix} \tag{148}$$

*will exist.*

*If this condition be satisfied we may find the limiting probability*

$$\pi(t;\varepsilon)=\lambda_{i_1 i_2 \cdots\cdots i_l},$$

*at the point* $(t_{1i_1}, t_{2i_2}, \cdots\cdots, t_{li_l})$ *by solving*

$$T_{m_1, m_2, \cdots\cdots, m_l}=\sum_{i_1=1}^{r_1}\sum_{i_2=1}^{r_2}\cdots\cdots\sum_{i_l=1}^{r_l} t_{1i_1}^{m_1} t_{2i_2}^{m_2}\cdots\cdots t_{li_l}^{m_l}\,.\,\lambda_{i_1 i_2 \cdots\cdots i_l}, \tag{149}$$

$$m_1=0, 1, 2, \cdots\cdots, r_1-1,$$
$$m_2=0, 1, 2, \cdots\cdots, r_2-1,$$
$$\cdots\cdots\cdots\cdots\cdots\cdots$$
$$m_l=0, 1, 2, \cdots\cdots, r_l-1,$$

*or by means of the relation*

$$\lambda_{i_1 i_2 \cdots\cdots i_l}=\frac{\sum_{m_1=0}^{r_1-1}\cdots\cdots\sum_{m_l=0}^{r_l-1} a_{m_1 m_2 \cdots\cdots m_l}\,.\,T_{m_1 m_2 \cdots\cdots m_l}}{f_{1i_1}(t_{1i_1})f_{2i_2}(t_{2i_2})\cdots\cdots f_{li_l}(t_{li_l})},$$

*where*

$$f_{1i_1}(t)=\frac{(t-t_{11})(t-t_{12})\cdots\cdots(t-t_{1r_1})}{(t-t_{1i_1})},$$

$$f_{2i_2}(t)=\frac{(t-t_{21})(t-t_{22})\cdots\cdots(t-t_{2r_2})}{(t-t_{2i_2})},$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

*and* $a_{m_1 m_2 \cdots\cdots m_l}$ *are to be determined from*

$$f_{1i_1}(t_1)f_{2i_2}(t_2)\cdots\cdots f_{li_l}(t_l)=\sum_{m_1=0}^{r_1-1}\cdots\cdots\sum_{m_l=0}^{r_l-1} a_{m_1 m_2 \cdots m_l}\,.\,t_1^{m_1} t_2^{m_2}\cdots\cdots t_l^{m_l}\,.$$

*The values of* $\lambda_{i_1 i_2 \cdots\cdots i_l}$ *so found may vanish, in which case the point*

$$(t_{1i_1}, t_{2i_2}, \cdots\cdots, t_{li_l})$$

*belongs to the set* $A_0$; *or may not vanish, in which case the point belongs to*

*the set $A_1$, so that the number of the non-vanishing $\lambda_{i_1 i_2 \cdots\cdots i_l}$ gives that of the points belonging to the set $A_1$.*

**26.** We now go on to the calculation of $T_{m_1, m_2, \cdots\cdots, m_l}$.

Let there be given $l$ mutually exclusive events $E_1, E_2, \cdots\cdots, E_l$ and let the occurrence of $E_i$ in the first trial be denoted by $H_i$, that in the second trial by $EH_i$, the concurrence of $E_i$ in the first and $E_j$ in the second trials by $H_i H_j$ and so on(1). If we expand

$$\Delta = \{E + H_1(x_1 - 1) + H_2(x_2 - 1) + \cdots\cdots + H_l(x_l - 1)\}^n,$$

by applying the fundamental laws of algebra *with the exception of the commutative laws among $E, H_1, H_2, \cdots\cdots, H_l$ in the product expressions,* we obtain

$$\Delta = \Sigma\, \overline{P}_{m_1 m_2 \cdots\cdots m_l}(n)(x_1 - 1)^{m_1}(x_2 - 1)^{m_2} \cdots\cdots (x_l - 1)^{m_l},$$

the summation being extended with respect to $m_1, m_2, \cdots\cdots, m_l$ over

$$0 \leqq m_1 + m_2 + \cdots\cdots + m_l \leqq n,$$

where

$$\overline{P}_{0,0,0,\cdots\cdots 0}(n) = E^n,$$
$$\overline{P}_{1,0,0,\cdots\cdots 0}(n) = H_1 + EH_1 + E^2 H_1 + \cdots\cdots + E^{n-1} H_1,$$
$$\overline{P}_{0,1,0,\cdots\cdots 0}(n) = H_2 + EH_2 + E^2 H_2 + \cdots\cdots + E^{n-1} H_2,$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

and in general $\overline{P}_{m_1, m_2, \cdots\cdots, m_l}(n)$ is a certain sum of the symbols above defined.

If now the sum of the probabilities of the configurations which occur in $\overline{P}_{m_1, m_2, \cdots\cdots, m_l}(n)$ be denoted by

$$P_{m_1, m_2, \cdots\cdots, m_l}(n),$$

then it can be shown(2) that

$$\Sigma\, F(n; s_1, s_2, \cdots\cdots, s_l)\, x_1^{s_1} x_2^{s_2} \cdots\cdots x_l^{s_l}$$
$$= \Sigma\, P_{m_1 m_2 \cdots\cdots m_l}(n)(x_1 - 1)^{m_1}(x_2 - 1)^{m_2} \cdots\cdots (x_l - 1)^{m_l},$$

where the summation on the left is to be performed for $s_1, s_2, \cdots\cdots, s_l$ satisfying

(1) Regarding the fundamental properties of these notations, reference should be made to my previous paper.

(2) The proof can be carried out in a quite similar manner as in my previous paper.

$$0 \leqq s_1 + s_2 + \cdots\cdots + s_l \leqq n.$$

Consequently we have, as in the article 7,

$$T_{m_1, m_2, \cdots\cdots, m_l} = \lim_{n=\infty} \sum_s \left(\frac{s_1}{n}\right)^{m_1} \left(\frac{s_2}{n}\right)^{m_2} \cdots\cdots \left(\frac{s_l}{n}\right)^{m_l} F(n; s_1, s_2, \cdots, s_l)$$

$$= m_1!\, m_2! \cdots\cdots m_l! \lim_{n=\infty} \frac{P_{m_1 m_2 \cdots\cdots m_l}(n)}{n^{m_1+m_2+\cdots\cdots+m_l}}, \qquad (150)$$

*a formula by aid of which* $T_{m_1, m_2, \cdots\cdots, m_l}$ *can be calculated.*

**27.** As an illustration of the preceding result let us consider an *example where three persons* $A, B$ *and* $C$ *play a set of games each of which can not be drawn and in which their respective chances of winning in the* $m^{\text{th}}$ *game are as exhibited in the following table:—*

| *A's chance* | *B's chance* | *C's chance* | |
|---|---|---|---|
| $p_1$ | $p_1'$ | $1-p_1-p_1'$ | *in the first game* |
| $P_m$ | $P_m'$ | $1-P_m-P_m'$ | *if A wins the game before* |
| $Q_m$ | $Q_m'$ | $1-Q_m-Q_m'$ | *B* ,, |
| $R_m$ | $R_m'$ | $1-R_m-R_m'$ | *C* ,, |

$P_m, Q_m, R_m$ *as well as* $P_m', Q_m', R_m'$ *being subjected to certain conditions.*

Let $p_m$ be the probability of $A$'s winning in the $m^{\text{th}}$ game *a priori* and $p_m'$ that of $B$; also let $W_{m_1, m_2}(1)$ be the probability that $A$ wins both in the $m_1{}^{\text{th}}$ and $m_2{}^{\text{th}}$ games, winning or losing in other games being disregarded, $W_{m_1, m_2}(2)$ that $A$ wins in the $m_1{}^{\text{th}}$ and $B$ in the $m_2{}^{\text{th}}$ games $(m_1 < m_2)$, $W_{m_1, m_2}(3)$ that $B$ wins in the $m_1{}^{\text{th}}$ and $A$ in the $m_2{}^{\text{th}}$, $W_{m_1, m_2}(4)$ that $B$ wins both in the $m_1{}^{\text{th}}$ and $m_2{}^{\text{th}}$ games and so on. Then if $s_1$ denotes the number of winning of $A$ in $n$ games and $s_2$ that of $B$, we have evidently

$$\sum_{0 \leqq s_1+s_2 \leqq n} F(n; s_1, s_2)\, x_1^{s_1} x_2^{s_2} = 1 + P_{10}(n)(x_1-1) + P_{01}(n)(x_2-1)$$
$$+ P_{20}(n)(x_1-1)^2 + P_{11}(n)(x_1-1)(x_2-1)$$
$$+ P_{02}(n)(x_2-1)^2 + \cdots\cdots, \qquad (151)$$

where

$$P_{10}(n) = p_1 + p_2 + \cdots\cdots + p_n, \qquad P_{01}(n) = p_1' + p_2' + \cdots\cdots + p_n'$$

$$P_{20}(n) = \sum_{m_2=2}^{n} \sum_{m_1=1}^{m_2-1} W_{m_1, m_2}(1), \qquad P_{02}(n) = \sum_{m_2=2}^{n} \sum_{m_1=1}^{m_2-1} W_{m_1, m_2}(4)$$

$$P_{11}(n) = \sum_{m_2=2}^{n} \sum_{m_1=1}^{m_2-1} \{ W_{m_1, m_2}(2) + W_{m_1, m_2}(3) \},$$

.....

Further it can be shown, as in § 8, that

$$p_m = p_{m-1}(P_m - R_m) + p'_{m-1}(Q_m - R_m) + R_m, \tag{152}$$

$$p_m' = p_{m-1}(P_m' - R_m') + p'_{m-1}(Q_m' - R_m') + R_m', \tag{153}$$

and also by putting

$$V_m(1) = \sum_{m_1=1}^{m-1} \{W_{m_1}, m(1) - p_{m_1} p_m\}, \quad V_m(2) = \sum_{m_1=1}^{m-1} \{W_{m_1}, m(2) - p_{m_1} p_m'\}$$

$$V_m(3) = \sum_{m_1=1}^{m-1} \{W_{m_1}, m(3) - p_{m_1}' p_m\}, \quad V_m(4) = \sum_{m_1=1}^{m-1} \{W_{m_1}, m(4) - p_{m_1}' p_m'\}$$

that

$$V_m(1) = (P_m - R_m) V_{m-1}(1) + (Q_m - R_m) V_{m-1}(2) + p_{m-1}(P_m - p_m), \tag{154}$$

$$V_m(2) = (Q_m' - R_m') V_{m-1}(2) + (P_m' - R_m') V_{m-1}(1) + p_{m-1}(P_m' - p_m'), \tag{155}$$

$$V_m(3) = (P_m - R_m) V_{m-1}(3) + (Q_m - R_m) V_{m-1}(4) + p'_{m-1}(Q_m - p_m), \tag{156}$$

$$V_m(4) = (Q_m' - R_m') V_{m-1}(4) + (P_m' - R_m') V_{m-1}(3) + p'_{m-1}(Q_m' - p_m'). \tag{157}$$

Now let us first consider *the case when*

$$\begin{aligned} &\lim P_m = P, \quad &&\lim Q_m = Q, \quad &&\lim R_m = R, \\ &\lim P_m' = P', \quad &&\lim Q_m' = Q', \quad &&\lim R_m' = R' \end{aligned} \tag{158}$$

*exist so as to satisfy*

$$\Delta = \begin{vmatrix} 1 - P + R & -Q + R \\ -P' + R' & 1 - Q' + R' \end{vmatrix} \neq 0. \tag{159}$$

In this case by putting

$$p_m - p_{m-1}(P - R) - p'_{m-1}(Q - R) = \rho_m,$$

$$p_m' - p'_{m-1}(Q' - R') - p_{m-1}(P' - R') = \rho_m',$$

we get from (152) and (153)

$$\lim \rho_m = R, \qquad \lim \rho_m' = R',$$

and therefore

$$(1 - P + R) \frac{\sum_1^n p_m}{n} - (Q - R) \frac{\sum_1^n p_m'}{n} \to R,$$

$$-(P'-R')\frac{\sum_1^n p_m}{n}+(1-Q'+R')\frac{\sum_1^n p_m'}{n}\to R',$$

which give

$$\frac{\sum_1^n p_m}{n}\to\frac{(1-Q')R+R'Q}{\Delta},\qquad \frac{\sum_1^n p_m'}{n}\to\frac{(1-P)R'+P'R}{\Delta}.$$

Consequently we have

$$T_{10}=p=\frac{(1-Q')R+R'Q}{\Delta},$$

$$T_{01}=p'=\frac{(1-P)R'+P'R}{\Delta}.$$

Again by (154) and (155) we get

$$(1-P+R)\frac{\sum_2^n V_m(1)}{n^2}-(Q-R)\frac{\sum_2^n V_m(2)}{n^2}\to 0,$$

$$-(P'-R')\frac{\sum_2^n V_m(1)}{n^2}+(1-Q'+R')\frac{\sum_2^n V_m(2)}{n^2}\to 0,$$

and accordingly

$$\frac{\sum_2^n V_m(1)}{n^2}\to 0 \text{ and } \frac{\sum_2^n V_m(2)}{n^2}\to 0.$$

Similarly from (156) and (157)

$$\frac{\sum_2^n V_m(3)}{n^2}\to 0 \text{ and } \frac{\sum_2^n V_m(4)}{n^2}\to 0.$$

We have therefore

$$T_{20}-p^2=2.\lim_{n=\infty}\frac{\sum_2^n V_m(1)}{n^2}=0,$$

$$T_{02}-p'^2=2.\lim_{n=\infty}\frac{\sum_2^n V_m(4)}{n^2}=0,$$

i. e. $$T_{20}=p^2,\qquad T_{02}=p'^2,$$

which leads us to the result:

*The set of games is of the first class with respect to the winning of A and B, having a single point* $(p, p')$ *of the first class.*

Next consider *the case when* $\Delta$ *vanishes with some additional conditions.*

Since $\Delta$ may be written in the form

$$\Delta = \{(1-P)(1-Q')-P'Q\}+(1-Q')R+R'Q+(1-P)R'+P'R,$$

and we have $1-P \geqq P'$, $1-Q' \geqq Q$, we see at once that $\Delta$ vanishes when and only when

$$(1-P)(1-Q')=P'Q, \quad (1-Q')R=0,$$
$$R'Q=0, \qquad (1-P)R'=0, \qquad P'R=0.$$

These give rise to the following six alternatives:—

| | | | |
|---|---|---|---|
| (i) | $P=1,\ Q'=1$ | (and accordingly | $P'=0,\ Q=0$ ) |
| (ii) | $P=1,\ 1-R-R'=1$ | ( ,, | $P'=0,\ R=0,\ R'=0$) |
| (iii) | $Q'=1,\ 1-R-R'=1$ | ( ,, | $Q=0,\ R=0,\ R'=0$) |
| (iv) | $P=1,\ Q=0,\ R=0$ | ( ,, | $P'=0$ ) |
| (v) | $P'=0,\ Q'=1,\ R'=0$ | ( ,, | $Q=0$ ) |
| (vi) | $1-P-P'=0, 1-Q-Q'=0, 1-R-R'=1$ | ( ,, | $R=0, R'=0$) |

of which the first three cases have the same type as well as the last three.

We shall here consider *the two cases* (i) *and* (iv) *confining ourselves to such special cases where in* (i)

$$m^a(1-P_m), \quad m^a(1-Q_m'), \quad m^a(R-R_m), \quad m^a(R'-R_m') \qquad (160)$$

*tend to four finite limits respectively, while in* (iv)

$$m^a(1-P_m), \qquad m^a Q_m, \qquad m^a R_m \qquad (161)$$

*tend to three finite limits, a being a positive number greater than 1 in both cases.*

1°. *Case* (i).

Since (152) may be written in the form

$$p_m - p_{m-1} = p'_{m-1} Q_m - p_{m-1}(1-P_m) + R_m(1-p_{m-1}-p'_{m-1})$$

we have

$$p_{m+l} - p_{m-1} = \sum_{k=m}^{m+l} \{p'_{k-1} Q_k - p_{k-1}(1-P_k)\} + \sum_{k=m}^{m+l} R_k(1-p_{k-1}-p'_{k-1}), \qquad (162)$$

of which the left-hand side is bounded for all $n$. If now the condition (160) be satisfied the series $\sum_{k=m}^{\infty} \{ p'_{k-1} Q_k - p_{k-1}(1-P_k)\}$ will be convergent and therefore by (162) the positive series $\sum_{k=m}^{\infty} R_k(1-p_{k-1}-p'_{k-1})$ can not be divergent and hence again by making use of the same formula we see that

$$\lim_{l=\infty} p_{m+l}=p$$

should exist. Similarly we may show the existence of

$$\lim_{m=\infty} p_m'=p'.$$

Consequently we have

$$T_{10}=p, \qquad T_{01}=p'. \tag{163}$$

Especially when $R$ or $R'$ does not vanish the relation (152) or (153) gives

$$p+p'=1. \tag{164}$$

Again since $V_m(i)/n$ $(i=1, 2, 3, 4)$ are bounded for all $n$, we have from (154), (155), (156) and (157), if the condition (160) be satisfied,

$$V_m(1)-V_{m-1}(1)+R\{V_{m-1}(1)+V_{m-1}(2)\}\to p(1-p), \tag{165}$$

$$V_m(2)-V_{m-1}(2)+R'\{V_{m-1}(1)+V_{m-1}(2)\}\to -pp', \tag{166}$$

$$V_m(3)-V_{m-1}(3)+R\{V_{m-1}(3)+V_{m-1}(4)\}\to -pp', \tag{167}$$

$$V_m(4)-V_{m-1}(4)+R'\{V_{m-1}(3)+V_{m-1}(4)\}\to p'(1-p'). \tag{168}$$

Hence if both of $R$ and $R'$ vanish we get

$$V_m(1)-V_{m-1}(1)\to p(1-p), \qquad V_m(2)-V_{m-1}(2)\to -pp',$$
$$V_m(3)-V_{m-1}(3)\to -pp', \qquad V_m(4)-V_{m-1}(4)\to p'(1-p'),$$

from which we arrive at

$$T_{20}=p, \qquad T_{11}=0, \qquad T_{02}=p'.$$

If, on the contrary, one at least of $R$ and $R'$ does not vanish, then since $1-p=p'$, we have, from (165) and (166), by adding side by side

$$\{V_m(1)+V_m(2)\}-\{V_{m-1}(1)+V_{m-1}(2)\}+(R+R')\{V_{m-1}(1)+V_{m-1}(2)\}\to 0,$$

from which we get, by Kubota's modification of Pólya's theorem(1),

$$V_m(1)+V_m(2)\to 0,$$

and, by making use again of the relations (165) and (166) respectively,

$$T_{20}=p, \qquad \frac{V_m(2)}{m}\to -pp'.$$

Similarly, from (167) and (168),

$$\frac{V_m(3)}{m}\to -pp', \qquad T_{02}=p'.$$

Thus we have, in all cases,

$$T_{20}=T_{10}, \qquad T_{11}=0, \qquad T_{02}=T_{01},$$

and consequently the relations (141) and (142) for $r_1=2$ and $r_2=2$ respectively are to be satisfied by

$$x_1=0, \quad x_2=1\,; \quad \lambda_1=1-p, \quad \lambda_2=p.$$
$$y_1=0, \quad y_2=1\,; \quad \lambda_1'=1-p', \quad \lambda_2'=p'.$$

Then we obtain, by solving the system of equations (143) for those values of $x_1, x_2, y_1$ and $y_2$,

$$\lambda_{11}=1-p-p', \quad \lambda_{12}=p', \quad \lambda_{21}=p, \quad \lambda_{22}=0.$$

We have therefore:

*The set of games is of the first class with respect to the winning of A and B having three points* $(0,0), (0,1), (1,0)$ *of the first class if no one of* $1-p-p'$, $p'$ *and* $p$ *vanish and having two points of the same class if one, but no two, of them vanish.*

2°. *Case* (iv).

If, in this case, $R'$ or $1-Q'$ vanish we have a special case of (ii) or (i) and therefore we shall here assume that $R'\neq 0$ and $1-Q'\neq 0$.

Now, just in the same way as in the above, we may show the existence of

$$\lim p_m=p,$$

and hence by (153) we obtain

$$\lim p_m'=p'=\frac{R'(1-p)}{1-Q'+R'}\,. \tag{169}$$

(1) L. c.

Again the relation (154) gives in this case

$$V_m(1)-V_{m-1}(1)\to p(1-p),$$

so that we get

$$T_{20}=p=T_{10}. \tag{170}$$

Similarly from (156)

$$\frac{V_m(3)}{m}\to -pp'.$$

The two relations (155) and (157) now give respectively

$$\lim_{m=\infty}\frac{V_m(2)}{m}=-\frac{R'}{1-Q'+R'}\cdot\lim_{m=\infty}\frac{V_m(1)}{m}$$

$$=-\frac{R'p(1-p)}{1-Q'+R'}=-pp',$$

$$\lim_{m=\infty}\frac{V_m(4)}{m}=-\frac{R'p}{1-Q'+R'}\cdot p'$$

and therefore

$$T_{11}=2pp'+\lim_{m=\infty}\frac{V_m(2)+V_m(3)}{m}=0, \tag{171}$$

$$T_{02}=p'^2+\lim_{m=\infty}\frac{V_m(4)}{m}=\frac{R'}{1-Q'+R'}p'. \tag{172}$$

The relation (170) shows us that the system of equations (141) for $r_1=2$ is consistent. But as the expression (172) for $T_{02}$ gives no indication as to the least $r_2$, unless $p=0$ or $p=1$, we have further to calculate $T_{03}, T_{04},\cdots\cdots$ in that case.

Now if we denote the probability that $B$ wins in the two games $m_1^{\text{th}}$ and $m_2^{\text{th}}$ and $A$ in the $m^{\text{th}}$ $(m_1<m_2<m)$ by $W_{m_1,m_2,m}(5)$, the probability that $B$ wins in the three games named by $W_{m_1,m_2,m}(6)$ and put

$$V_m(5)=\sum_{m_2=2}^{m-1}\sum_{m_1=1}^{m_2-1}\{W_{m_1,m_2,m}(5)-W_{m_1,m_2}(4)\cdot p_m\},$$

$$V_m(6)=\sum_{m_2=2}^{m-1}\sum_{m_1=1}^{m_2-1}\{W_{m_1,m_2,m}(6)-W_{m_1,m_2}(4)\cdot p_m'\},$$

we shall obtain

$$V_m(5)=(P_m-R_m)\,V_{m-1}(5)+(Q_m-R_m)\,V_{m-1}(6)$$
$$+(Q_m-p_m)\sum_{m_1=1}^{m-2} W_{m_1,\,m-1}(4), \qquad (173)$$

$$V_m(6)=(Q_m'-R_m')\,V_{m-1}(6)+(P_m'-R_m')\,V_{m-1}(5)$$
$$+(Q_m'-p_m')\sum_{m_1=1}^{m-2} W_{m_1,\,m-1}(4). \qquad (173)'$$

These two relations give, in the case considered,

$$\lim_{m=\infty}\frac{V_m(5)}{m^2}=-\frac{p}{2}\,T_{02},$$

$$\lim_{m=\infty}\frac{V_m(6)}{m^2}=-\frac{R'}{1-Q'+R'}\cdot\lim_{m=\infty}\frac{V_m(5)}{m^2}=\frac{R'p}{1-Q'+R'}\cdot\frac{T_{02}}{2}$$

and we have ultimately, by aid of (169),

$$T_{03}=\frac{R'}{1-Q'+R'}\,T_{02}.$$

In a quite similar manner we may derive

$$T_{04}=\frac{R'}{1-Q'+R'}\,T_{03},$$

so that the system of equations (142) for $r_2=2$ will have a unique solution

$$y_1=0, \qquad y_2=\frac{R'}{1-Q'+R'}, \qquad \lambda_1'=1-p', \quad \lambda_2'=p'.$$

Then we obtain, by solving the system of equations (143) for the values of $x_1, x_2, y_1, y_2$ thus found and making use of the relation (169),

$$\lambda_{11}=0, \qquad \lambda_{12}=1-p, \qquad \lambda_{21}=p, \qquad \lambda_{22}=0.$$

We have therefore:

*The set of games is of the first class with respect to the winning of A and B having two points* $\left(0,\ \dfrac{R'}{1-Q'+R'}\right)$ *and* $(1,0)$ *of the first class provided that* $p\neq 0$ *and* $p\neq 1$.

**28.** Before leaving the present Part we shall express the results obtained in § 25 and § 26 in another form, in conformity with that for a more general case which will be considered in the next Part. The transformed result runs as follows:—

*In order that a system of trials may be of the first class with respect to a system of $l$ mutually exclusive events it is necessary and sufficient that there should exist a positive integer $r$ such that the system of equations*

$$T_{m_1 m_2 \cdots \cdots m_l} = \sum_{i=1}^{r} t_{1i}^{m_1} \, t_{2i}^{m_2} \cdots\cdots t_{li}^{m_l} \, . \, \lambda_i \text{ (1)}$$

$$m_1 = 0, 1, 2, \cdots\cdots, 2r,$$

$$m_2 = 0, 1, 2, \cdots\cdots, 2r,$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$m_l = 0, 1, 2, \cdots\cdots, 2r$$

*will have a unique solution.*

If this condition be satisfied, then the set $A_1$ will consist of $r$ points whose coordinates are

$$t_{1i}, t_{2i}, \cdots\cdots, t_{li} \qquad (i = 1, 2, 3, \cdots\cdots, r),$$

and at which the limiting probabilities are given by

$$\pi(t; \varepsilon) = \lambda_i, \qquad (i = 1, 2, \cdots\cdots, r).$$

From this it follows also that:

*In order that a system of trials may be of the first class having $r$ points of the first class with respect to a system of $l$ mutually exclusive events it is necessary and sufficient that the function $\phi(t_1, t_2, \cdots\cdots, t_l)$ defined by*

$$\phi(t_1, t_2, \cdots\cdots, t_l) = \sum_{m_1=0}^{\infty} \cdots\cdots \sum_{m_l=0}^{\infty} T_{m_1 m_2 \cdots\cdots m_l} \, t_1^{m_1} \, t_2^{m_2} \cdots\cdots t_l^{m_l}$$

*is a rational function of $t_1, t_2, \cdots\cdots, t_l$ which is expressible in the form*

$$\phi(t_1, t_2, \cdots\cdots, t_l) = \sum_{i=1}^{r} \frac{\lambda_i}{(1 - t_{1i} t_1)(1 - t_{2i} t_2) \cdots\cdots (1 - t_{li} t_l)} \, .$$

$$\lambda_i \neq 0 \qquad (i = 1, 2, \cdots\cdots, r)$$

---

(1) Here some of $t_{1i}$ $(i = 1, 2, \cdots, r)$ may be equal to one another as well as some of $t_{2i}$, &c; the meaning attached to $t_{1i}, t_{2i}, \cdots\cdots$ differs from that given in § 25.

## Part V.

### Determinate System of Trials with respect to a System of Mutually Exclusive Events.

**29.** We now proceed further to consider a more general system of trials with respect to a system of mutually exclusive events.

Let $s_1, s_2, \cdots\cdots, s_l$ denote the numbers of the happening of $l$ mutually exclusive events $E_1, E_2, \cdots\cdots, E_l$ in $n$ trials such that

$$s_1+s_2+\cdots\cdots+s_l \leqq n,$$

and let $\pi_n(\alpha, \beta)$ denote the probability that $s_1, s_2, \cdots\cdots, s_l$ will satisfy simultaneously

$$\alpha_1 \leqq \frac{s_1}{n} \leqq \beta_1, \quad \alpha_2 \leqq \frac{s_2}{n} \leqq \beta_2, \cdots\cdots\cdots, \alpha_l \leqq \frac{s_l}{n} \leqq \beta_l.$$

Then the limit $\pi(\alpha, \beta)$ to which $\pi_n(\alpha, \beta)$ tends when $n=\infty$, is a certain function of $\alpha(\alpha_1, \alpha_2, \cdots\cdots, \alpha_l)$ and $\beta(\beta_1, \beta_2, \cdots\cdots, \beta_l)$, if it exist.

Now a point $t=(t_1, t_2, \cdots\cdots, t_l)$ within the closed domain $D$ bounded by the coordinate planes

$$t_i=0, \qquad (i=1, 2, \cdots\cdots, l),$$

and a plane $t_1+t_2+\cdots\cdots+t_l=1$ in $l$-dimensional space may be such that $\pi(t-\varepsilon, t+\varepsilon')$ exists at least for points $\varepsilon=(\varepsilon_1, \varepsilon_2, \cdots\cdots, \varepsilon_l)$ and $\varepsilon'=(\varepsilon_1', \varepsilon_2', \cdots\cdots, \varepsilon_l')$, where $\varepsilon_1, \varepsilon_2, \cdots\cdots, \varepsilon_l, \varepsilon_1', \varepsilon_2', \cdots\cdots, \varepsilon_l'$ denote positive numbers belonging to $2l$ positive sets $S_1, S_2, \cdots\cdots, S_l, S_1', S_2', \cdots\cdots, S_l'$ respectively, which have the point 0 as their common limiting point. For such a point $t$, if

$$\overline{\lim_{n=\infty}}\ \pi_n(t-\bar{\varepsilon}, t+\bar{\varepsilon}') \text{ and } \underline{\lim_{n=\infty}}\ \pi_n(t-\bar{\varepsilon}, t+\bar{\varepsilon}'),$$

both of which necessarily exist for *all* positive $\bar{\varepsilon}=(\bar{\varepsilon}_1, \bar{\varepsilon}_2, \cdots\cdots, \bar{\varepsilon}_l)$ and $\bar{\varepsilon}'=(\bar{\varepsilon}_1', \bar{\varepsilon}_2', \cdots\cdots, \bar{\varepsilon}_l')$ and also are monotonic functions of them, be denoted by

$$\overline{\pi}(t-\bar{\varepsilon}, t+\bar{\varepsilon}') \quad \text{and} \quad \underline{\pi}(t-\bar{\varepsilon}, t+\bar{\varepsilon}')$$

respectively, we have

$$\lim_{\bar{\varepsilon}, \bar{\varepsilon}'=+0} \overline{\pi}(t-\bar{\varepsilon}, t+\bar{\varepsilon}') = \lim_{\bar{\varepsilon}, \bar{\varepsilon}'=+0} \underline{\pi}(t-\bar{\varepsilon}, t+\bar{\varepsilon}') = \lambda,$$

where $\lambda$ is a definite number between 0 and 1 and therefore

$$\lim_{\varepsilon, \varepsilon'=+0} \pi(t-\varepsilon,\ t+\varepsilon')=\lambda,$$

provided that $\varepsilon_1, \varepsilon_2, \cdots\cdots, \varepsilon_l, \varepsilon_1', \varepsilon_2', \cdots\cdots, \varepsilon_l'$ *belong respectively to* $S_1, S_2, \cdots\cdots, S_l, S_1', S_2', \cdots\cdots, S_l'$.

Let us call such a point $t$ the *point at which the limiting probability* $\pi(t-0, t+0)$ *is determinate* and is equal to $\lambda$, and let the two sets of points at which

$$\lambda=0 \quad \text{and} \quad \lambda\neq 0$$

respectively be denoted by $B_0$ and $B_1$. It is evident that the set $B_1$ is an enumerable set, so that the complementary set $C(B_1)$ in the domain $D$ has the same measure as the domain $D$.

If now $t$ be a point of the set $B_0$ it is easy to see that

$$\pi(t, t)=0.$$

Again it can be shown that *if* $\pi(0, t)$, *where*

$$0=(0, 0, \cdots\cdots, 0), \qquad t=(t_1, t_2, \cdots\cdots, t_l),$$

*be continuous in the domain consisting of an everywhere dense set of points in the l-dimensional space*

$$0\leqq t_i\leqq 1, \qquad (i=1, 2, 3, \cdots\cdots, l), \tag{174}$$

*then every point within the domain D belongs either to the set* $B_0$ *or to the set* $B_1$.

We shall show this for the case $l=2$ ([1]).

Let $\pi(a, b; x, y)$ denote the limiting probability $\pi(a, t)$ where

$$a=(a, b), \qquad t=(x, y),$$

i.e. the limit of the probability $\pi(a, b; x, y)$, when $n=\infty$, that $s_1, s_2$, the numbers of the happening of two mutually exclusive events $E_1, E_2$ in $n$ trials, will satisfy simultaneously

$$a\leqq\frac{s_1}{n}\leqq x, \qquad b\leqq\frac{s_2}{n}\leqq y;$$

and suppose that the function $\pi(0, 0; x, y)$ be continuous in a domain

([1]) Throughout the present Part we shall consider this special case, the reasoning however being perfectly general.

$\overline{D}'$ consisting of an everywhere dense set of points in the domain $\overline{D}$ defined by

$$0 \leqq x \leqq 1, \qquad 0 \leqq y \leqq 1 \text{ (}^1\text{)}.$$

Then if $(x_1, y_1)$ be a point belonging to $\overline{D}'$ we may take 4 positive numbers $\varepsilon_1, \varepsilon_2, \varepsilon_1'$ and $\varepsilon_2'$ corresponding to an arbitrarily chosen positive number $\delta$ so as to satisfy

$$\pi\,(0, 0;\ x_1+\varepsilon_1',\ y_1+\varepsilon_2') - \pi\,(0, 0;\ x_1-\varepsilon_1,\ y_1-\varepsilon_2) < \delta.$$

From this we obtain at once

$$\pi\,(0, y_1;\ x_1, y_1) = 0,$$
$$\pi\,(x_1, 0;\ x_1, y_1) = 0,$$

and also if $\overline{\pi}\,(a, b; x, y)$ and $\underline{\pi}\,(a, b; x, y)$ denote the maximum and the minimum limits of $\pi_n\,(a, b; x, y)$ when $n$ becomes infinite respectively, then

$$\lim_{\varepsilon,\, \varepsilon' = +0} \overline{\underline{\pi}}\,(0, y_1-\varepsilon;\ x_1, y_1+\varepsilon') = 0,$$

and

$$\lim_{\varepsilon,\, \varepsilon' = +0} \overline{\underline{\pi}}\,(x_1-\varepsilon, 0;\ x_1+\varepsilon', y_1) = 0.$$

By these relations we may show that, if $(x_1, y_1)$ be a point belonging to the everywhere dense set $\overline{D}'$ in the domain $\overline{D}$, then the segment

$$0 \leqq x \leqq x_1, \qquad y = y_1$$

as well as the segment

$$x = x_1, \qquad 0 \leqq y \leqq y_1$$

contains an everywhere dense set, having the linear measure equal to the length of the interval, of points at which the function $\pi\,(0, t)$ exists and its extended function $\overline{\lambda}\,(t)$ is continuous.

For by taking a point $(X, y_1)$ where $0 \leqq X \leqq x_1$, if we choose four positive numbers $\varepsilon_1, \varepsilon_2, \varepsilon_1', \varepsilon_2'$ from four positive sequences converging to zero respectively such that the two points

$$(X+\varepsilon_1', y_1+\varepsilon_2') \quad \text{and} \quad (X-\varepsilon_1, y_1-\varepsilon_2)$$

always belong to $\overline{D}'$, then

---

(1) It can easily be shown that the function is continuous in an everywhere dense set of points in any domain such that $0 \leqq x$, $0 \leqq y$, in this case.

$$\pi(0,0;\ X+\varepsilon_1', y_1+\varepsilon_2')-\pi(0,0;\ X-\varepsilon_1, y_1-\varepsilon_2)$$

exists and is a monotonic function of $\varepsilon_1, \varepsilon_2, \varepsilon_1', \varepsilon_2'$ belonging to the sequences, so that it has a determinate limit $\lambda$ when

$$\varepsilon_1=+0, \quad \varepsilon_2=+0, \quad \varepsilon_1'=+0, \quad \varepsilon_2'=+0.$$

Now, by the property just proved,

$$\lim_{\varepsilon_2, \varepsilon_2'=+0} \overline{\pi}(0, y_1-\varepsilon_2;\ X, y_1+\varepsilon_2')=0,$$

and hence we get

$$\lim_{\varepsilon_1, \varepsilon_1'=+0} \overline{\pi}(X-\varepsilon_1, 0;\ X+\varepsilon_1', y_1)=\lambda.$$

It is clear that the set of points $X$ in $(0, x_1)$ such that the corresponding $\lambda$ is different from zero is an enumerable set in that interval, and therefore we must have

$$\lambda=0,$$

at points $X$ forming a set with the measure equal to the length of the interval $(0, x_1)$. It follows that the points $(X, y_1)$ at which

$$\lim_{\varepsilon_1', \varepsilon_2'=+0} \pi(0,0;\ X+\varepsilon_1', y_1+\varepsilon_2')=\lim_{\varepsilon_1, \varepsilon_2=+0} \pi(0,0;\ X-\varepsilon_1, y_1-\varepsilon_2), \tag{175}$$

i.e. the extended function of $\pi(0,0;\ x, y)$ is continuous (1) form an everywhere dense set in the segment

$$0 \leqq x \leqq x_1, \quad y=y_1,$$

whose measure is equal to the length of the interval $(0, x_1)$.

Now by the results just obtained it is easy to show the theorem enunciated. Let $(\bar{x}, \bar{y})$ be any point within the domain $D$ bounded by the coordinate axes

$$x=0, \quad y=0,$$

and the line $x+y=1$. Then after taking three positive numbers $\varepsilon_1, \varepsilon_1'$, $\varepsilon_2'$ such that $(\bar{x}-\varepsilon_1, \bar{y}+\varepsilon_2')$ and $(\bar{x}+\varepsilon_1', \bar{y}+\varepsilon_2')$ belong to the points of continuity of $\pi(0,0;\ x, y)$ we may choose, in virtue of the fact that the points of continuity on two parallel segments

$$x=\bar{x}-\varepsilon_1, \quad 0 \leqq y \leqq \bar{y}+\varepsilon_2',$$

(1) For the function $\pi(0,0;x,y)$ is monotonic, i.e.

$$\pi(0,0;\ x, y) \geqq \pi(0,0;\ \bar{x}, \bar{y}) \quad \text{if} \quad x \geqq \bar{x} \quad \text{and} \quad y \geqq \bar{y}.$$

and

$$x=\bar{x}+\varepsilon_1', \quad 0 \leqq y \leqq \bar{y}+\varepsilon_2',$$

form two sets whose linear measures are identical with the lengths of the segments respectively, another positive number $\varepsilon_2$ such that $(\bar{x}-\varepsilon_1, \bar{y}-\varepsilon_2)$ and $(\bar{x}+\varepsilon_1', \bar{y}-\varepsilon_2)$ are also points of continuity of $\pi(0, 0; x, y)$.

Now for such positive numbers $\varepsilon_1, \varepsilon_2, \varepsilon_1', \varepsilon_2'$ we obtain the existence of

$$\begin{aligned}\tau = &\{\pi(0, 0; \bar{x}+\varepsilon_1', \bar{y}+\varepsilon_2') - \pi(0, 0; \bar{x}+\varepsilon_1', \bar{y}-\varepsilon_2)\} \\ &- \{\pi(0, 0; \bar{x}-\varepsilon_1, \bar{y}+\varepsilon_2') - \pi(0, 0; \bar{x}-\varepsilon_1, \bar{y}-\varepsilon_2)\}, \qquad (176)\end{aligned}$$

which is the limit of the probability, when $n=\infty$, that $s_1, s_2$ will satisfy simultaneously

$$\bar{x}-\varepsilon_1 < \frac{s_1}{n} \leqq \bar{x}+\varepsilon_1', \quad \bar{y}-\varepsilon_2 < \frac{s_2}{n} \leqq \bar{y}+\varepsilon_2',$$

and as we have

$$\pi(x-\varepsilon_1, 0; \bar{x}-\varepsilon_1, \bar{y}+\varepsilon_2') = 0,$$
$$\pi(0, \bar{y}-\varepsilon_2; \bar{x}+\varepsilon_1', \bar{y}-\varepsilon_2) = 0,$$

i.e. the limit of the probability, when $n=\infty$, that $s_1, s_2$ will satisfy simultaneously

$$\bar{x}-\varepsilon_1 = \frac{s_1}{n}, \quad 0 \leqq \frac{s_2}{n} \leqq \bar{y}+\varepsilon_2',$$

or

$$0 \leqq \frac{s_1}{n} \leqq \bar{x}+\varepsilon_1', \quad \bar{y}-\varepsilon_2 = \frac{s_2}{n},$$

vanishes, it follows that $\pi(\bar{x}-\varepsilon_1, \bar{y}-\varepsilon_2; \bar{x}+\varepsilon_1', \bar{y}+\varepsilon_2')$ exists and is equal to $\tau$. This always holds for $\varepsilon_1, \varepsilon_2, \varepsilon_1', \varepsilon_2'$ belonging to four positive sets which have the point 0 as their common limiting point, and hence we have

$$\lim_{\varepsilon_1, \varepsilon_2, \varepsilon_1', \varepsilon_2' = +0} \pi(\bar{x}-\varepsilon_1, \bar{y}-\varepsilon_2; \bar{x}+\varepsilon_1', \bar{y}+\varepsilon_2') = \lambda,$$

where $\lambda$ is a definite number, zero or positive,— a result showing that the point $(\bar{x}, \bar{y})$ belongs to the set $B_0$ or to the set $B_1$.

We can further prove that if $\pi(0, 0; x, y)$ be continuous in a domain $D'$ consisting of an everywhere dense set of points in $D$ then it is also continuous in the domain consisting of an everywhere dense set of points in $\bar{D}$, and consequently we have the result:

*If $\pi(0,t)$ be continuous within the domain $D'$ consisting of an everywhere dense set of points in the domain $D$ defined by*

$$0 \leqq x, \quad 0 \leqq y, \quad x+y \leqq 1,$$

*then every point within the domain $D$ belong either to the set $B_0$ or to the set $B_1$*( [1] ).

It seems to me, but I have not succeeded to show at length, that the converse theorem does not necessarily hold, i. e. from the existence of the two sets $B_0$ and $B_1$ only in the domain $D$ can not be derived the existence and the continuity of the function $\pi(0,t)$ at an everywhere dense set of points in the domain $\bar{D}$.

In the case when $\pi(0,0;x,y)$ is continuous in the domain $D$ and accordingly in the whole domain $\bar{D}$, the expression $\tau$ defined by (176) vanishes in the limit when $\varepsilon_1, \varepsilon_2, \varepsilon_1', \varepsilon_2' = +0$ and hence we have:

*If the function $\pi(0,t)$ be continuous in the domain $D$, then every point within the same domain belongs to the set $B_0$.*

That the converse is not necessarily true can at once be seen if we observe that $\lim\limits_{\varepsilon_1, \varepsilon_2, \varepsilon_1', \varepsilon_2' = +0} \tau$ vanishes, for example, in case when $\pi(0,0;1,y)$ is a continous function of $y$ in the interval $(0 \leqq y \leqq 1)$ or $\pi(0,0;x,1)$ is a continuous function of $x$ in the interval $(0 \leqq x \leqq 1)$ $\pi(0,0;x,y)$ being a continuous function of $x$ and $y$ in the domain consisting of an everywhere dense set but not in the whole domain $D$.

For example if

$$\pi(0,0;x,y) = 0 \qquad \text{in} \quad 0 \leqq x < p, \quad 0 \leqq y \leqq 1-x,$$

$$= \frac{y}{1-p} \qquad \text{in} \quad p < x \leqq 1, \quad 0 \leqq y \leqq 1-x,$$

then every point within the domain $D$ belongs to the set $B_0$; still $\pi(0,t)$ is discontinuous at points on the line $x=p$( [2] ).

In the present Part we have to deal with the case for which the function $\pi(0,t)$ is continuous in the domain $D'$ consisting of an everywhere dense set of points in $D$ and therefore its extended function $\bar{\lambda}(t)$ is continuous and identical with the limiting probability $\pi(0,t)$ at points in the domain $\bar{D}'$, having the measure equal to that of the domain $\bar{D}$, which includes $D'$ as a component and is contained in $\bar{D}$.

---

( [1] ) The assumption that $\pi(0,t)$ is continuous in the domain consisting of an everywhere dense set of points in $D$ may be replaced by that $\pi(0,t)$ exists at points of a set everywhere dense in $D$, to arrive at the same result.

( [2] ) The actual occurrence of such a case will be considered in § 35.

Such a system of trials is called the *determinate system* with respect to the system of mutually exclusive events under consideration.

**30.** We shall now find conditions for the occurrence of a determinate system of trials with respect to a system of two mutually exclusive events.

If a system of trials belong to a determinate system with respect to a system of two mutually exclusive events, then it has been seen that the function $\pi(0, 0; x, y)$, i. e. $\pi(0, t)$ exists and is continuous in the domain $\bar{D}'$ which is contained in $\bar{D}$ and has the measure equal to it, where $\bar{D}$ is defined by

$$0 \leqq x \leqq 1, \quad 0 \leqq y \leqq 1,$$

and also that if $(x_1, y_1)$ belong to $\bar{D}'$, in this case, the segment

$$0 \leqq x \leqq x_1, \quad y = y_1,$$

as well as the segment

$$x = x_1, \quad 0 \leqq y \leqq y_1$$

contains an everywhere dense set, having the linear measure equal to the length of the segment respectively, of points at which the function $\pi(0, t)$ is continuous.

Consequently in case when the point $(1, 1)$ belongs to the set $\bar{D}'$, the points of continuity, on the two lines

$$0 \leqq x \leqq 1, \quad y = 1,$$

and

$$x = 1, \quad 0 \leqq y \leqq 1,$$

of the extended function $\bar{\lambda}(t) = \bar{\lambda}(x, y)$ of $\pi(0, 0; x, y)$ form two everywhere dense sets respectively, both having the linear measure equal to unity; and in case when the point $(1, 1)$ does not belong to $\bar{D}'$, by taking two positive numbers $\varepsilon, \varepsilon'$ we obtain the two sets lying on the segments

$$0 \leqq x \leqq 1, \quad y = 1 + \varepsilon',$$

and

$$x = 1 + \varepsilon, \quad 0 \leqq y \leqq 1,$$

respectively, whose linear measures are equal to unity and at each point of which the function $\bar{\lambda}(t)$ is continuous.

Let the projections, on the $x$- and the $y$-axes, of the two sets of points thus obtained be denoted by $J_x$ and $J_y$ respectively so that $J_x$ and $J_y$ form

two linear sets each having the linear measure equal to unity, and let the complementary sets of $J_x$ and $J_y$ in the respective segments $0 \leqq x \leqq 1$, $y=0$ and $x=0$, $0 \leqq y \leqq 1$ be denoted by $I_x$ and $I_y$.

It is evident that if $x=x_1$ belong to $J_x$ or $y=y_1$ belong to $J_y$, the point $(x_1, y_1)$ belongs to the set $B_0$, and if $x=x_1$ belong to $I_x$ while $y=y_1$ belongs to $I_y$ then the point $(x_1, y_1)$ belongs to either $B_1$ or $B_1$.

Now by dividing the intervals $(0 \leqq x \leqq 1)$ and $(0 \leqq y \leqq 1)$ into $k$ and $l$ sub-intervals of which the end points $X_r$ $(r=0, 1, 2, \cdots\cdots, k)$ and $Y_{r'}$, $(r'=0, 1, 2, \cdots\cdots, l)$ belong, except at least $X_0=0$, $X_k=1$ and $Y_0=0$, $Y_l=1$, to the sets $J_x$ and $J_y$ respectively we have, if $m_1, m_2 \geqq 0$,

$$\sum_{r=1}^{k} \sum_{r'=1}^{l} X_{r-1}^{m_1} Y_{r'-1}^{m_2} \pi (X_{r-1}, Y_{r'-1}; X_r, Y_{r'})$$

$$\leqq \varlimsup_{n=\infty} \sum_{s_1+s_2 \leqq n} \left(\frac{s_1}{n}\right)^{m_1} \left(\frac{s_2}{n}\right)^{m_2} F(n; s_1, s_2), \qquad (177)$$

and

$$\sum_{r=1}^{k} \sum_{r'=1}^{l} X_r^{m_1} Y_{r'}^{m_2} \pi (X_{r-1}, Y_{r'-1}; X_r, Y_{r'})$$

$$\geqq \varlimsup_{n=\infty} \sum_{s_1+s_2 \leqq n} \left(\frac{s_1}{n}\right)^{m_1} \left(\frac{s_2}{n}\right)^{m_2} F(n; s_1, s_2). \qquad (177)'$$

The two expressions standing on the left of (177) and (177)′ respectively tend to a common limit

$$1-\int_0^1 \frac{d}{dx}(x^{m_1}) \cdot \bar{\lambda}(x, 1)\, dx - \int_0^1 \frac{d}{dy}(y^{m_2}) \cdot \bar{\lambda}(1, y)\, dy$$

$$+\int_0^1\int_0^1 \frac{d}{dx}(x^{m_1}) \frac{d}{dy}(y^{m_2}) \cdot \bar{\lambda}(x, y)\, dx\, dy\,(^1), \qquad (178)$$

when the subdivision advances without limit such that the greatest sub-intervals have the limit zero, and hence we get

$$T_{m_1, m_2} = \int x^{m_1} y^{m_2}\, d\bar{\lambda}, \qquad (179)$$

where $T_{m_1, m_2}$ stands for

$$\lim_{n=\infty} \sum_{s_1+s_2 \leqq n} \left(\frac{s_1}{n}\right)^{m_1} \left(\frac{s_2}{n}\right)^{m_2} F(n; s_1, s_2),$$

---

(1) When $y=1$ is a line of discontinuity of $\bar{\lambda}(x, y)$, $\bar{\lambda}(x, 1)$ is to be taken in the sense of $\bar{\lambda}(x, 1+0)$. A similar remark applies to $\bar{\lambda}(1, y)$.

and

$$\int f_1(x) f_2(y)\, d\bar{\lambda}$$

for the expression

$$f_1(1) f_2(1) - f_2(1) \int_0^1 f_1'(x)\, \bar{\lambda}(x, 1)\, dx - f_1(1) \int_0^1 f_2'(y)\, \bar{\lambda}(1, y)\, dy$$
$$+ \int_0^1 \int_0^1 f_1'(x) f_2'(y)\, \bar{\lambda}(x, y)\, dx\, dy. \qquad (180)$$

It follows therefore that if a system of trials be of a determinate one with respect to a system of two mutually exclusive events, then there should exist a monotonic function $\bar{\lambda}(x, y)$ satisfying (179).

Conversely *if there exist a limited and non-negative monotonic function* $\bar{\lambda}(x, y)$ *satisfying (179) for* $m_1 = 0, 1, 2, \cdots\cdots$, $m_2 = 0, 1, 2, \cdots\cdots$, *then we may show that, at points of continuity of* $\bar{\lambda}(x, y)$,

$$\pi(0, 0; x, y) = \bar{\lambda}(x, y).$$

To show this we establish the following lemma :—

If $g_1(x)$ and $g_2(x)$ be any continuous functions of $x$ in the interval $(0, 1)$ such that their derivatives $g_1'(x)$ and $g_2'(x)$ are also continuous in the same interval, then

$$\lim_{n=\infty} \sum_{s_1+s_2 \leqq n} g_1\left(\frac{s_1}{n}\right) g_2\left(\frac{s_2}{n}\right) F(n; s_1, s_2) = \int g_1(x)\, g_2(y)\, d\bar{\lambda}, \qquad (181)$$

in case when $\bar{\lambda}(x, y)$ satisfies the relations (179).

It will easily be seen that this relation holds good if $g_1(x)$ and $g_2(x)$ are both polynomials of $x$.

If, on the contrary, one or both of $g_1(x)$ and $g_2(x)$ are not polynomials of $x$ then choose two polynomials $f_1'(x)$ and $f_2'(x)$ so as to satisfy

$$\left.\begin{array}{l} |f_1'(x) - g_1'(x)| < \delta \\ |f_2'(x) - g_2'(x)| < \delta \end{array}\right\} \quad \text{for} \quad 0 \leqq x \leqq 1,$$

where $\delta$ is an arbitrarily chosen positive number(1).

For these polynomials $f_1'(x)$ and $f_2'(x)$ we get easily, if $0 \leqq x \leqq 1$,

$$\left| \int_0^x f_1'(x)\, dx - \int_0^x g_1'(x)\, dx \right| < \delta.$$

(1) This is always possible by Weierstrass' theorem as $g_1'(x)$ and $g_2'(x)$ are assumed to be continuous in the interval $(0, 1)$.

$$\left|\int_0^x f_2'(x)\,dx-\int_0^x g_2'(x)\,dx\right|<\delta\,,$$

and therefore by taking suitably the arbitrary constants which enter in $f_1(x)$ and $f_2(x)$ respectively, we have

$$\left.\begin{array}{l}|f_1(x)-g_1(x)|<\delta\\ |f_2(x)-g_2(x)|<\delta\end{array}\right\}\quad\text{for}\quad 0\leqq x\leqq 1\,.$$

Consequently if $M$ be the greatest of the greatest moduli which the four functions $g_1(x)$, $g_2(x)$, $g_1'(x)$ and $g_2'(x)$ assume in the interval $(0,1)$ then

$$\begin{array}{l}|f_1(x)f_2(y)-g_1(x)\,g_2(y)|<\delta\,(2M+\delta),\\ |f_1(x)f_2'(y)-g_1(x)\,g_2'(y)|<\delta\,(2M+\delta),\\ |f_1'(x)f_2(y)-g_1'(x)\,g_2(y)|<\delta\,(2M+\delta),\\ |f_1'(x)f_2'(y)-g_1'(x)\,g_2'(y)|<\delta\,(2M+\delta),\end{array}$$

for $0\leqq x\leqq 1$ and $0\leqq y\leqq 1$.

We have therefore

$$\left|\overline{\lim_{n=\infty}}\sum_{s_1+s_2\leqq n} g_1\left(\frac{s_1}{n}\right)g_2\left(\frac{s_2}{n}\right)F(n;s_1,s_2)\right.$$

$$\left.-\lim_{n=\infty}\sum_{s_1+s_2\leqq n} f_1\left(\frac{s_1}{n}\right)f_2\left(\frac{s_2}{n}\right)F(n;s_1,s_2)\right|<\delta\,(2M+\delta),$$

$$\left|\int g_1(x)\,g_2(y)\,d\bar{\lambda}-\int f_1(x)f_2(y)\,d\bar{\lambda}\right|$$

$$\leqq\delta\,(2M+\delta)\left\{1+\int_0^1\bar{\lambda}(x,1)\,dx+\int_0^1\bar{\lambda}(1,y)\,dy+\int_0^1\!\!\int_0^1\bar{\lambda}(x,y)\,dx\,dy\right\},$$

These two relations, combined with

$$\lim_{n=\infty}\sum_{s_1+s_2\leqq n} f_1\left(\frac{s_1}{n}\right)f_2\left(\frac{s_2}{n}\right)F(n;s_1,s_2)=\int f_1(x)f_2(y)\,d\bar{\lambda}\,,$$

give rise to

$$\left|\overline{\lim_{n=\infty}}\sum_{s_1+s_2\leqq n} g_1\left(\frac{s_1}{n}\right)g_2\left(\frac{s_2}{n}\right)F(n;s_1,s_2)-\int g_1(x)\,g_2(y)\,d\bar{\lambda}\right|$$

$$\leqq\delta\,(2M+\delta)\left\{2+\int_0^1\bar{\lambda}(x,1)\,dx+\int_0^1\bar{\lambda}(1,y)\,dy+\int_0^1\!\!\int_0^1\bar{\lambda}(x,y)\,dx\,dy\right\}.$$

The positive number $\delta\,(2M+\delta)$ of which the left-hand side expression

is quite independent, may be taken as small as we please, so that we arrive at

$$\lim_{n=\infty} \sum_{s_1+s_2\leq n} g_1\left(\frac{s_1}{n}\right) g_2\left(\frac{s_2}{n}\right) F(n; s_1, s_2) = \int g_1(x)\, g_2(y)\, d\bar{\lambda}\,,$$

thus establishing the lemma completely.

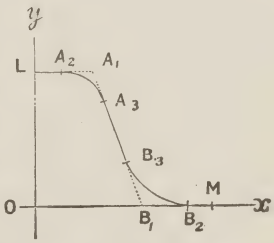
We now apply the lemma to the functions $g_1(x)$ and $g_2(x)$ defined as follows:—

Let $L$ and $M$ be two points $(0, 1)$ and $(1, 0)$. Further let $A_2$ and $A_1$ be the two points, on the line $y=1$, whose abscissae are respectively $a-\varepsilon_1$ and $a$ where

$$0<a-\varepsilon_1<a<1\,,$$

and let $B_1, B_2$ be the two points, on the $x$-axis, whose abscissae are respectively $b$ and $b+\varepsilon_2$ where

$$a<b<b+\varepsilon_2<1\,.$$




Draw two circular arcs $A_2A_3$ and $B_3B_2$, one touching the two lines $A_1A_2$ and $A_1B_1$, and another touching $B_1B_2$ and $B_1A_1$. Then we define $g_1(x)$ to be such a function of $x$ that its graph is represented by the full line $L\,A_2\,A_3\,B_3\,B_2\,M$.

Again let $L'$ and $M'$ be two points $(0, 1)$ and $(1, 0)$. Further let $A_2'$ and $A_1'$ be the two points, on the line $y=1$, whose abscissae are respectively $a'-\varepsilon_1'$ and $a'$ where

$$0<a'-\varepsilon_1'<a'<1\,,$$

and let $B_1', B_2'$ be the two points, on the $x$-axis, whose abscissae are respectively $b'$ and $b'+\varepsilon_2'$ where

$$a'<b'<b'+\varepsilon_2'<1\,.$$

Draw two circular arcs $A'_2\,A_3'$ and $B_3'\,B_2'$, one touching the two lines $A_1'\,A_2'$ and $A_1'\,B_1'$, and another touching $B_1'\,B_2'$ and $B_1'\,A_1'$. Then we define $g_2(x)$ to be such a function of $x$ that its graph is represented by the line $L'\,A_2'\,A_3'\,B_3'\,B_2'\,M'$.

Since for the functions $g_1(x)$ and $g_2(x)$ thus defined we have

$$\int g_1(x)\, g_2(y)\, d\bar{\lambda} = \int_{a-\varepsilon_1}^{b+\varepsilon_2} \int_{a'-\varepsilon_1'}^{b'+\varepsilon_2'} g_1'(x)\, g_2'(y)\, \bar{\lambda}(x, y)\, dx\, dy\,,$$

the relation (181) gives us

$$\lim_{n=\infty} \sum_{\substack{s_1 \leqq n(b+\varepsilon_2) \\ s_2 \leqq n(b'+\varepsilon_2')}} g_1\left(\frac{s_1}{n}\right) g_2\left(\frac{s_2}{n}\right) F(n; s_1, s_2)$$

$$= \int_{a-\varepsilon_1}^{b+\varepsilon_2} \int_{a'-\varepsilon_1'}^{b'+\varepsilon_2'} g_1'(x)\, g_2'(y)\, \bar{\lambda}(x, y)\, dx\, dy. \qquad (182)$$

Now let the function to which $g_1(x)$ tends when $\varepsilon_1, \varepsilon_2$ vanish, $a$ and $b$ remaining constant, i.e. the function represented by the broken line $LA_1B_1M$ be denoted by $\bar{g}_1(x)$; and similarly let the function to which $g_2(x)$ tends when $\varepsilon_1', \varepsilon_2'$ vanish be denoted by $\bar{g}_2(x)$. Then since the convergence of $g_1(x)$ and $g_2(x)$ to $\bar{g}_1(x)$ and $\bar{g}_2(x)$ is uniform we have easily

$$\lim_{\substack{\varepsilon_1, \varepsilon_2 \\ \varepsilon_1', \varepsilon_2' = 0}} \left[ \lim_{n=\infty} \sum_{\substack{s_1 \leqq n(b+\varepsilon_2) \\ s_2 \leqq n(b'+\varepsilon_2')}} g_1\left(\frac{s_1}{n}\right) g_2\left(\frac{s_2}{n}\right) F(n; s_1, s_2) \right]$$

$$= \lim_{n=\infty} \sum_{\substack{s_1 \leqq nb \\ s_2 \leqq nb'}} \bar{g}_1\left(\frac{s_1}{n}\right) \bar{g}_2\left(\frac{s_2}{n}\right) F(n; s_1, s_2).$$

On the other hand if $c, c'$ denote the trigonometrical tangents of the angles which $A_1B_1$ and $A_1'B_1'$ make with the axis of $x$ with their signs changed respectively, i. e.

$$c = \frac{1}{b-a}, \qquad c' = \frac{1}{b'-a'}, \qquad (183)$$

then we have always, on the circular arcs of the graph,

$$|g_1'(x)| \leqq c, \qquad |g_2'(x)| \leqq c',$$

independent of $\varepsilon_1, \varepsilon_2, \varepsilon_1', \varepsilon_2'$ so that by putting

$$\int_{a-\varepsilon_1}^{b+\varepsilon_2} \int_{a'-\varepsilon_1'}^{b'+\varepsilon_2'} g_1'(x)\, g_2'(y)\, \bar{\lambda}(x, y)\, dx\, dy$$

$$- cc' \int_a^b \int_{a'}^{b'} \bar{\lambda}(x, y)\, dx\, dy = \rho(\varepsilon_1, \varepsilon_2, \varepsilon_1', \varepsilon_2'),$$

we also arrive at

$$\lim_{\varepsilon_1, \varepsilon_2, \varepsilon_1', \varepsilon_2' = 0} \rho(\varepsilon_1, \varepsilon_2, \varepsilon_1', \varepsilon_2') = 0.$$

It follows therefore, by (182), that

$$\lim_{n=\infty} \sum_{\substack{s_1 \leqq nb \\ s_2 \leqq nb'}} \bar{g}_1\left(\frac{s_1}{n}\right)\bar{g}_2\left(\frac{s_2}{n}\right) F(n\,;\, s_1, s_2)$$

$$= cc' \int_a^b \int_{a'}^{b'} \bar{\lambda}(x, y)\, dx\, dy \text{ (1)}. \qquad (184)$$

By aid of this relation we may show our theorem for the case in which the point $(x, y)$ lies in the open domain $0<x<1$, $0<y<1$.

For from the relation (184) we get

$$\overline{\lim_{n=\infty}} \sum_{\substack{0 \leqq \frac{s_1}{n} \leqq a \\ 0 \leqq \frac{s_2}{n} \leqq a'}} F(n\,;\, s_1, s_2) \leqq \frac{1}{(b-a)(b'-a')} \int_a^b \int_{a'}^{b'} \bar{\lambda}(x, y)\, dx\, dy, \qquad (185)$$

$$\underline{\lim_{n=\infty}} \sum_{\substack{0 \leqq \frac{s_1}{n} \leqq b \\ 0 \leqq \frac{s_2}{n} \leqq b'}} F(n\,;\, s_1, s_2) \geqq \frac{1}{(b-a)(b'-a')} \int_a^b \int_{a'}^{b'} \bar{\lambda}(x, y)\, dx\, dy. \qquad (186)$$

The left-hand side expression of (185) is independent of $b$ and $b'$ while the right-hand side expression tends to $\bar{\lambda}(a, a')$ when $b$ and $b'$ converge to $a$ and $a'$ respectively, provided that $\bar{\lambda}(x, y)$ be continuous at the point $x=a$, $y=a'$. We have therefore

$$\overline{\lim_{n=\infty}} \sum_{\substack{0 \leqq \frac{s_1}{n} \leqq a \\ 0 \leqq \frac{s_2}{n} \leqq a'}} F(n\,;\, s_1, s_2) \leqq \bar{\lambda}(a, a'). \qquad (187)$$

Similarly from (186), if $\bar{\lambda}(x, y)$ be continuous at $x=b$, $y=b'$, then

$$\underline{\lim_{n=\infty}} \sum_{\substack{0 \leqq \frac{s_1}{n} \leqq b \\ 0 \leqq \frac{s_2}{n} \leqq b'}} F(n\,;\, s_1, s_2) \geqq \bar{\lambda}(b, b'). \qquad (188)$$

The two relations (187) and (188) show us that

(1) This shows that the lemma above established also holds for functions $g_1'(x)$ nad $g_2'(x)$ which are not necessarily continuous.

$$\lim_{n=\infty} \sum_{\substack{0 \leqq \frac{s_1}{n} \leqq x \\ 0 \leqq \frac{s_2}{n} \leqq y}} F(n; s_1, s_2) = \pi(0, 0; x, y)$$

exists and is equal to $\bar{\lambda}(x, y)$ if

$$0 < x < 1, \qquad 0 < y < 1,$$

and $\bar{\lambda}(x, y)$ be continuous at the point $(x, y)$.

Again to show our theorem for any point on the boundary of the rectangle $(0 \leqq x \leqq 1,\ 0 \leqq y \leqq 1)$, say for the point $x=1$, $y=a'$ where

$$0 < a' < 1,$$

we have to take

$$g_1(x) = 1,$$

in the whole interval $(0, 1)$, the function $g_2(x)$ being defined as above. Then since $g_1(x)=1$, $g_1'(x)=0$ in the whole interval we have

$$\int g_1(x)\, g_2(x)\, d\bar{\lambda} = -\int_{b'}^{a'} g_2'(x)\, \bar{\lambda}(1, y)\, dy,$$

and therefore we get, instead of (184),

$$\lim_{n=\infty} \sum_{\substack{0 \leqq \frac{s_1}{n} \leqq 1 \\ 0 \leqq \frac{s_2}{n} \leqq b'}} \bar{g}_2\left(\frac{s_2}{n}\right) F(n; s_1, s_2) = c' \int_{a'}^{b'} \bar{\lambda}(1, y)\, dy,$$

where $\bar{g}_2(x)$ and $c'$ have the same meaning as in the above. From this we obtain

$$\lim_{n=\infty} \sum_{\substack{0 \leqq \frac{s_1}{n} \leqq 1 \\ 0 \leqq \frac{s_2}{n} \leqq a'}} F(n; s_1, s_2) = \bar{\lambda}(1, a'), \tag{189}$$

provided that $\bar{\lambda}(x, y)$ is continuous at the point $x=1$, $y=a'$. Similar results hold for any point on the other boundary lines of the rectangle $(0 \leqq x \leqq 1,\ 0 \leqq y \leqq 1)$.

It has thus been shown that if there exist a limited and non-negative monotonic function $\bar{\lambda}(x, y)$ satisfying (179) for $m_1 = 0, 1, 2, \cdots\cdots$, $m_2 = 0, 1,$

2, ·····, then, at any point of continuity of $\bar{\lambda}(x, y)$,(1)

$$\pi(0, 0 : x, y) = \bar{\lambda}(x, y),$$

so that the system of trials belongs to a determinate system with respect to a system of two mutually exclusive events under consideration.

**31.** Just in the same manner as in § 17 it will be shown that, if

$$T_{m_1, m_2} \begin{pmatrix} m_1 = 0, 1, 2, \cdots\cdots \\ m_2 = 0, 1, 2, \cdots\cdots \end{pmatrix}$$

all exist, then there always exists a limited and non-negative monotonic function $\bar{\lambda}(x, y)$ satisfying a system of integral equations (179).

This, combined with the results obtained in the last article, gives the following criterion :—

*In order that a system of trials may be a determinate system with respect to a system of two mutually exclusive events it is necessary and sufficient that all* $T_{m_1 m_2}$ $(m_1 = 0, 1, 2, \cdots\cdots, m_2 = 0, 1, 2, \cdots\cdots)$ *should exist.*

*If all* $T_{m_1 m_2}$ *exist, then there always exists a non-negative monotonic function* $\bar{\lambda}(x, y)$ *satisfying a system of integral equations*

$$T_{m_1 m_2} = \int x^{m_1} y^{m_2} \, d\bar{\lambda}, \tag{179}$$

$$m_1 = 0, 1, 2, \cdots\cdots, \quad m_2 = 0, 1, 2, \cdots\cdots,$$

*and this function* $\bar{\lambda}(x, y)$ *is identical with the limiting probability* $\pi(0, 0; x, y)$ *at points of its continuity in the domain* $\bar{D}$ *defined by*

$$0 \leqq x \leqq 1, \qquad 0 \leqq y \leqq 1,$$

*the points forming a set whose measure is equal to the area of the domain* $\bar{D}$.

Let us now transform the system of integral equations (179) in a somewhat different form.

If the set of points at which the monotonic function $\bar{\lambda}(x, y)$ defined by (179) is continuous be denoted by $\bar{B}_0$(2) so that $\bar{B}_0$ is a component of the set $B_0$ where $B_0$ has been defined in § 29, then the function $\Psi(x, y)$ defined at points $(x, y)$ of the set $\bar{B}_0$ by,

$$\Psi(x, y) = \bar{\lambda}(x, y) - w(x, y),$$

---

(1) Of course the set of points of continuity of $\bar{\lambda}(x, y)$ is everywhere dense in the domain $0 \leqq x \leqq 1$, $0 \leqq y \leqq 1$.

(2) $\bar{B}_0$ is thus such a set of points that its projection on the $x$-axis is identical with the set $J_x$ defined in § 30 and its projection on the $y$-axis with the set $J_y$.

where $w(X, Y)$ denotes the sum of the limiting probabilities at points of the set $B_1$ in the domain $(0 \leqq x \leqq X,\ 0 \leqq y \leqq Y)$, will be continuous in the domain consisting of points of the set $\bar{B}_0$. Let $\Psi(x, y)$ and $W(x, y)$ be the extended functions of $\Psi(x, y)$ and $w(x, y)$ from their values at points of the set $\bar{B}_0$ in the extended domain which coincides with the entire domain $\bar{D}\,(0 \leqq x \leqq 1,\ 0 \leqq y \leqq 1)$. Then $\Psi(x, y)$ is a monotonic and connected(1) (but not necessarily continuous) function of $x$ and $y$ in the whole domain and satisfies, at every point of the set $\bar{B}_0$,

$$\Psi(x, y) = \bar{\lambda}(x, y) - W(x, y)$$

so that the relation (179) may be written in the form

$$T_{m_1 m_2} = \int x^{m_1} y^{m_2}\, d_1 \Psi + \int x^{m_1} y^{m_2}\, d_1 W, \tag{190}$$

where $\int x^{m_1} y^{m_2}\, d_1 \Psi$ and $\int x^{m_1} y^{m_1}\, d_1 W$ are defined by

$$\int x^{m_1} y^{m_2}\, d_1 \Psi$$

$$= \lim \sum_{r=1}^{k} \sum_{r'=1}^{l} X_r^{m_1}\, Y_{r'}^{m_2}\, \{\Psi(X_r, Y_{r'}) - \Psi(X_{r-1}, Y_{r'-1})\}, \tag{191}$$

$$\int x^{m_1} y^{m_2}\, d_1 W$$

$$= \lim \sum_{r=1}^{k} \sum_{r'=1}^{l} X_r^{m_1}\, Y_{r'}^{m_2}\, \{W(X_r, Y_{r'}) - W(X_{r-1}, Y_{r'-1})\}, \tag{192}$$

the limits being taken when

the greatest of $X_r - X_{r-1} \to 0$,

and ,, $Y_{r'} - Y_{r'-1} \to 0$, respectively

or by

$$\int x^{m_1} y^{m_2}\, d_1 \Psi = \Psi(1, 1) - \int_0^1 \frac{d}{dx}(x^{m_1})\, \Psi(x, 1)\, dx - \int_0^1 \frac{d}{dy}(y^{m_2})\, \Psi(1, y)\, dy$$
$$+ \int_0^1 \int_0^1 \frac{d}{dx}(x^{m_1}) \frac{d}{dy}(y^{m_2})\, \Psi(x, y)\, dx\, dy,$$

and a similar expression for $\int x^{m_1} y^{m_2}\, d_1 W$.

Now if $\lambda_1, \lambda_2, \lambda_3, \cdots\cdots$ denote the limiting probabilities $\pi\,(x-0, y-0;$

(1) A function $\Psi(x, y)$ of two variables $x$ and $y$ is said to be connected when if any two points be taken on the surface represented by $z = \Psi(x, y)$ there exists at least a line on the surface connecting those two points.

$x+0, y+0)$ at points $(x_1, y_1), (x_2, y_2), (x_3, y_3), \cdots\cdots$ of the set $B_1$ arranged in some order, then it will easily be shown that

$$\int x^{m_1} y^{m_2} d_1 W = \sum_{i=1}^{\infty} x_i^{m_1} y_i^{m_2} . \lambda_i ,$$

and hence the relation (190) is transformed into

$$T_{m_1 m_2} = \sum_{i=1}^{\infty} x_i^{m_1} y_i^{m_2} \lambda_i + \int x^{m_1} y^{m_2} d_1 \Psi, \tag{193}$$

$$m_1 = 0, 1, 2, \cdots\cdots, \quad m_2 = 0, 1, 2, \cdots\cdots .$$

We have therefore:

If all $T_{m_1 m_2}$ exist then the relations (193) must be satisfied for a connected function $\Psi(x, y)$ and for a set of sets of values of $(x_i, y_i, \lambda_i)$ $(i=1, 2, 3, \cdots\cdots)$ with the conditions

$$\left.\begin{array}{c} 0 \leqq x_i \leqq 1, \qquad 0 \leqq y_i \leqq 1, \\ \lambda_i \geqq 0 \text{ (}^1\text{)}. \end{array}\right\} \tag{194}$$

Conversely if these exist a connected function $\Psi(x, y)$ in the domain $\bar{D}$ $(0 \leqq x \leqq 1,\ 0 \leqq y \leqq 1)$ and a set of sets of values of $(x_i, y_i, \lambda_i)$ $(i=1, 2, 3, \cdots\cdots)$, subject to the conditions (194), so as to satisfy (193), then we can see at once that, at points of an everywhere dense set whose projections on the coordinate axes form the complementary sets to $(x_i)$ and $(y_i)$ respectively,

$$\pi(0, 0; x, y) = \Psi(x, y) + \Sigma_{xy} \lambda_i ,$$

where $\Sigma_{xy}$ denotes the summation about $\lambda_i$ corresponding to the points $(x_i, y_i)$ which lie in the domain $\{(0, x);\ (0, y)\}$.

Consequently we have the result:

*If $T_{m_1 m_2}$ $(m_1=0, 1, 2, \cdots\cdots; m_2=0, 1, 2, \cdots\cdots)$ exist, then the system of equations (193) is always satisfied by a connected function $\Psi(x, y)$ and a set of values of $(x_i, y_i, \lambda_i)$ subject to the conditions (194). The set of points $(x_i, y_i)$ corresponding to non-vanishing $\lambda_i$, in this case, forms the set $B_1$.*

**32.** From the analogy of the results obtained in § 18 and § 31 we may classify the determinate systems of trials with respect to a system of two mutually exclusive events, in a similar manner as in the determinate systems with respect to a single event.

The case when $\Psi(x, y)$ vanishes identically throughout the whole domain $\bar{D}$ $(0 \leqq x \leqq 1,\ 0 \leqq y \leqq 1)$ will be characterized by the condition

(1) In case when there exists none or a finite number of points of the set $B_1$, all or all but a finite number of $\lambda_i$ are to vanish.

$$T_{m_1 m_2} = \sum_{i=1}^{\infty} x_i^{m_1} y_i^{m_2} \lambda_i . \tag{195}$$

$$m_1 = 0, 1, 2, \cdots\cdots \quad m_2 = 0, 1. 2, \cdots\cdots$$

Such a system of trials is called the *simple determinate system.*

Again if $G$ denote the common measure of the set $B_1$ and its derivative then $G$ is a closed set and it may be resolved into a sum of an enumerable set $H$ and a perfect set $H_1$ of which the former can be expressed in the form

$$H = G_0 + G_1 + \cdots\cdots + G_\mu ,$$

where $\mu$ is a finite ordinal number or a transfinite ordinal number of the second class and $G_i$ denotes the isolated set of points contained in the $i^{\text{th}}$ derivative of $G$. We call the point belonging to $G_i$ the point of the $i+1^{\text{th}}$ class and the system of trials for which every point in the set $G$ is at most of the $i+1^{\text{th}}$ class the *system of trials of the* $i+1^{\text{th}}$ *class* or *of the generalized* $i+1^{\text{th}}$ *class* according as $\Psi(x, y)$ does or does not vanish identically. It is evident that for the occurrence of such a system of trials it is necessary and sufficient that the $i^{\text{th}}$ derivative of the set of points $(x_j, y_j)$ where $(x_1, y_1), (x_2, y_2), \cdots\cdots$ are as to satisfy the system of relations (195) or (193) should consist of a finite number of points( [1] ).

Furthermore if the set $G$ contains a perfect set $H_1$ as a component the system of trials is said to be of the *second kind,* and in this case it may have nowhere dense set of points of condensation of the relative frequency or everywhere dense set of points of its condensation in some sub-domain of $D\,(0 \leqq x,\ 0 \leqq y,\ x+y \leqq 1)$.

It will be observed that if a system of trials belong to a determinat system with respect to a system of two mutually exclusive events then the system of trials also belongs to a determinate system with respect to either one of the events. But in this case the class to which the system of trials belongs with respect to a system of two mutually exclusive events is not necessarily identical with the class to which the same system of trials belongs with respect to either one of the events. For example a system of trials which is of the generalized zero class with respect to two mutually exclusive events may be of the higher class with respect to both of the events respectively.

**33.** We have, in the Part IV, considered a very special case of the determinate system with respect to a system of two mutually exclusive events, the condition of whose occurrence can be expressed by some

( [1] ) In case when the set $H$ does not exist, the system of trials belongs to the zero class (generalized), thus answering the case when $i = -1$.

relations among a finite number of $T_{m_1, m_2}$, and in the present Part, its most general case whose occurrence depends entirely on the existence of all $T_{m_1 m_2}$ $(m_1=0, 1, 2, \cdots\cdots;\ m_2=0, 1, 2, \cdots\cdots)$, $\infty^2$ in number. We shall here consider a special case of the determinate system which may be characterized by some, $\infty^1$ in number, of $T_{m_1, m_2}$. It is the case when the system of trials is of the first class with respect to either one of the events.

If a determinate system of trials with respect to a system of two mutually exclusive events belong to a simple determinate system of the first class with respect to the first event, then from the discussions in § 5 and § 30, we see at once that:

(i) *The system of simultaneous equations*

$$T_{m_1, 0}=\sum_{i=1}^{r_1} x_i^{m_1} \lambda_i, \qquad m_1=0, 1, 2, \cdots\cdots, 2r_1, \tag{196}$$

*will have a unique solution, for a properly chosen positive number* $r_1$, *and*

$$\text{(ii)} \qquad T_{m_1, m_2} \quad \begin{pmatrix} m_1=0, 1, \cdots\cdots r_1-1 \\ m_2=0, 1, \cdots\cdots \inf \end{pmatrix}$$

*necessarily exist.*

We shall now show that *these two conditions are sufficient to ensure the occurrence of a determinate system with respect to a system of two mutually exclusive events, which is of the first class with respect to the first event.*

If $T_{m_1, m_2}$ for a definite $m_1$ and for all $m_2 (m_2=0, 1, 2, \cdots\cdots)$ exist, then it can be shown, as in § 17, that there exists a monotonic function

$$\bar{\lambda}^{(m_1)}(y)$$

of $y$ satisfying a system of integral equations

$$T_{m_1, m_2}=\int y^{m_2}\, d\bar{\lambda}^{(m_1)}, \tag{197}$$

$$m_2=0, 1, 2. 3, \cdots\cdots$$

where $\int f(y)\, d\bar{\lambda}^{(m_1)}$ is defined by

$$\int f(y)\, d\bar{\lambda}^{(m_1)}=f(1)\, \bar{\lambda}^{(m_1)}(1)-\int_0^1 f'(y)\, \bar{\lambda}^{(m_1)}(y)\, dy.$$

For such functions $\bar{\lambda}^{(m_1)}(y)$ $(m_1=0, 1, 2, \cdots\cdots, r_1-1)$, let us construct a series of functions $\bar{\lambda}_1(y), \bar{\lambda}_2(y), \cdots\cdots, \bar{\lambda}_{r_1}(y)$, so as to satisfy

$$\bar{\lambda}^{(m_1)}(y)=\sum_{i=1}^{r_1} x_i^{m_1}\,.\,\bar{\lambda}_i(y), \tag{198}$$

$$m_1=0, 1, 2, \cdots\cdots, r_1-1,$$

$x_1, x_2, \cdots\cdots, x_{r_1}$ being the constants which satisfy the simultaneous equations (196). That the functions $\lambda_i(y)$ $(i=1, 2, \cdots\cdots, r_1)$ are uniquely determined if the condition (i) be satisfied can at once be seen; and if they exist, they will be expressed in the linear form of monotonic functions $\bar{\lambda}^{(0)}(y), \bar{\lambda}^{(1)}(y), \cdots\cdots, \bar{\lambda}^{(r_1-1)}(y)$ so that they are integrable in the interval $(0, 1)$. The relation (197) then gives

$$T_{m_1 m_2}=\sum_{i=1}^{r_1} x_i^{m_1}\int y^{m_2}\, d\bar{\lambda}_i$$

$$m_2=0, 1, 2, \cdots\cdots .$$

Starting from these relations and following the reasoning adopted in § 30 we may derive, instead of (184),

$$\lim_{n=\infty} \sum_{\substack{0\leqq\frac{s_1}{n}\leqq 1\\ 0\leqq\frac{s_2}{n}\leqq b'}} f_1\left(\frac{s_1}{n}\right)\bar{g}_2\left(\frac{s_2}{n}\right)F(n; s_1, s_2)$$

$$=\frac{1}{c'}\sum_{i=1}^{r_1} f_1(x_i)\int_{a'}^{b'}\bar{\lambda}_i(y)\,dy,$$

where $f_1(x)$ is a polynomial of $x$ at most of the degree $r_1-1$, and $\bar{g}_2(y)$, $a', b', c'$ have the same meaning as in § 30; and therefore we have ultimately

$$\lim_{n=\infty} \sum_{\substack{0\leqq\frac{s_1}{n}\leqq 1\\ 0\leqq\frac{s_2}{n}\leqq y}} f_1\left(\frac{s_1}{n}\right)F(n; s_1, s_2)=\sum_{i=1}^{r_1} f_1(x_i)\,\bar{\lambda}_i(y), \tag{199}$$

at points $y$ of continuity of the function $\bar{\lambda}_i(y)$ i.e. of all the functions $\bar{\lambda}^{(0)}(y), \bar{\lambda}^{(1)}(y), \cdots\cdots \bar{\lambda}^{(r_1-1)}(y)$.

Since, if the condition (i) be satisfied all the points in the domain $D$ $(0\leqq x,\ 0\leqq y,\ x+y\leqq 1)$, whose abscissae are not equal to any one of $x_1, x_2, \cdots\cdots, x_{r_1}$, belong to the zero class (defined in § 23), if we take

$$f_1(x)=\frac{(x-x_1)(x-x_2)\cdots\cdots(x-x_{r_1})}{(x-x_i)},$$

for the function $f_1(x)$ in the relation (199), we shall obtain

$$\lim_{n=\infty} \sum_{\substack{x_i-\varepsilon \leqq \frac{s_1}{n} \leqq x_i+\varepsilon \\ 0 \leqq \frac{s_2}{n} \leqq y}} \left(\frac{s_1}{n}-x_1\right)\left(\frac{s_1}{n}-x_2\right)\cdots\cdots\left(\frac{s_1}{n}-x_{i-1}\right)\left(\frac{s_1}{n}-x_{i+1}\right)\cdots\cdots \times\left(\frac{s_1}{n}-x_{r_1}\right) F(n; s_1, s_2)$$

$$=(x_i-x_1)(x_i-x_2)\cdots\cdots(x_i-x_{i-1})(x_i-x_{i+1})\cdots\cdots (x_i-x_{r_1})\,\bar{\lambda}_i(y),$$

where $\varepsilon$ is a positive number which may be chosen arbitrarily small. It follows, as in § 3, that

$$\lim_{n=\infty} \sum_{\substack{\left|\frac{s_1}{n}-x_i\right| \leqq \varepsilon \\ 0 \leqq \frac{s_1}{n} \leqq y}} F(n; s_1, s_2)=\bar{\lambda}_i(y)\ (^1), \tag{200}$$

and therefore if $x_1, x_2, x_3, \cdots\cdots, x_{r_1}$ denote $(x_i)$ arranged in order of increasing magnitude then for the point $x$ such that

$$x_\mu < x < x_{\mu+1},$$

we obtain

$$\pi(0, 0; x, y)=\sum_{i=1}^{\mu} \bar{\lambda}_i(y),$$

provided that $y$ is a point of continuity of all functions $\bar{\lambda}_i(y)$ $(i=1, 2\cdots\cdots, \mu)$; and for $0 \leqq x < x_1$

$$\pi(0, 0; x, y)=0,$$

while for $x_{r_1} < x \leqq 1$

$$\pi(0, 0; x, y)=\sum_{i=1}^{r_1} \bar{\lambda}_i(y).$$

Thus $\pi(0, 0; x, y)$ has been shown to be continuous at an everywhere dense set of points in $D$ and hence the system of trials belongs to a determinate system with respect to a system of two mutually exclusive events under consideration.

The results may thus be stated as follows:—

*In order that a system of trials which is of the first class with respect to the first event, may be a determinate system with respect to a system of two mutually exclusive events it is necessary and sufficient that the two conditions* (i) *and* (ii) *should be satisfied.*

---

(1) It has indirectly been shown that $\bar{\lambda}_i(y)$ is a monotonic function of $y$ in the domain consisting of points of its continuity, and as the set of such points is everywhere dense in the interval $0 \leqq y \leqq 1$, it follows that $\bar{\lambda}_i(y)$ is a monotonic function in the whole interval $0 \leqq y \leqq 1$.

*If these conditions be satisfied, then there exists a system of monotonic functions* $\bar{\lambda}_1(y), \bar{\lambda}_2(y), \cdots\cdots, \bar{\lambda}_{r_1}(y)$ *satisfying* (198) *where* $\bar{\lambda}^{(0)}(y), \bar{\lambda}^{(1)}(y), \cdots\cdots, \bar{\lambda}^{(r_1-1)}(y)$ *are defined by* (197); *and at points* $y$ *of continuity of these monotonic functions we have the relation* (200).

**34.** The reasoning given in the foregoing articles may equally be applied to the general case and we have the results:—

In order that a system of trials belongs to a determinate system with respect to a system of $l$ mutually exclusive events it is necessary and sufficient that all $T_{m_1, m_2, \cdots, m_l}$

$$\begin{pmatrix} m_1=0, 1, 2, \cdots\cdots \\ m_2=0, 1, 2, \cdots\cdots \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ m_l=0, 1, 2, \cdots\cdots \end{pmatrix}$$

should exist.

If all $T_{m_1, m_2, \cdots, m_l}$ exist, then there always exists a monotonic function

$$\bar{\lambda}(t)=\bar{\lambda}(t_1, t_2, \cdots\cdots, t_l)$$

of $l$ variables $t_1, t_2, \cdots\cdots, t_l$ satisfying a system of integral equations

$$T_{m_1 m_2 \cdots m_l}=\int t_1^{m_1} t_2^{m_2} \cdots\cdots t_l^{m_l} d\bar{\lambda},$$

$$m_1=0, 1, 2, \cdots\cdots \qquad m_2=0, 1, 2, \cdots\cdots$$

$$\cdots\cdots\cdots\cdots\cdots\cdots \qquad m_l=0, 1, 2, \cdots\cdots$$

where $\int t_1^{m_1} t_2^{m_2} \cdots\cdots t_l^{m_l} d\bar{\lambda}$ is defined by

$$\lim_{\delta=0} \sum_{r_1=1}^{k_1} \sum_{r_2=1}^{k_2} \cdots\cdots \sum_{r_l=1}^{k_l} t_{1r_1}^{m_1} t_{2r_2}^{m_2} \cdots\cdots t_{lr_l}^{m_l} \Big\{ \bar{\lambda}(t_{1,r_1}, t_{2,r_2}, \cdots\cdots t_{l,r_l})$$

$$-\bar{\lambda}(t_{1,r_1-1}, t_{2,r_2-1}, \cdots\cdots t_{l,r_l-1}) \Big\}, \qquad (201)$$

$\delta$ denoting the greatest of

$$t_{1,r_1}-t_{1,r_1-1}, t_{2,r_2}-t_{2,r_2-1}, \cdots\cdots t_{l,r_l}-t_{l,r_l-1},$$

such that $\qquad t_{10}=t_{20}=\cdots\cdots=t_{l0}=0,$

and $\qquad t_{1k_1}=t_{2k_2}=\cdots\cdots=t_{lk_l}=1.$

This function $\bar{\lambda}(t)$ is identical with the limiting probability

$$\pi(0, t)=\pi(0, 0, \cdots\cdots 0, t_1, t_2, \cdots\cdots, t_l),$$

at points of its continuity in the domain $\overline{D}$ defined by

$$0 \leqq t_i \leqq 1, \qquad (i=1, 2, \cdots\cdots, l),$$

the points forming a set whose measure is equal to that of the domain $\overline{D}$. Or if all $T_{m_1, m_2, \cdots, m_l}$ exist then there exists a connected function

$$\Psi(t) = \Psi(t_1, t_2, \cdots\cdots, t_l),$$

and a set of values of $(t_{1i}, t_{2i}, \cdots\cdots t_{li}, \lambda_i)$ subject to the conditions

$$0 \leqq t_{1i} \leqq 1, \qquad 0 \leqq t_{2i} \leqq 0, \cdots\cdots, 0 \leqq t_{li} \leqq 1,$$

$$\lambda_i \geqq 0,$$

satisfying a system of relations

$$T_{m_1 m_2 \cdots m_l} = \sum_{i=1}^{\infty} t_{1i}^{m_1} t_{2i}^{m_2} \cdots\cdots t_{li}^{m_l} \lambda_i + \int t_1^{m_1} t_2^{m_2} \cdots\cdots l_l^{m_l} d, \Psi$$

$$m_1 = 0, 1, 2, \cdots\cdots \qquad m_2 = 0, 1, 2, \cdots\cdots$$

$$\cdots\cdots\cdots\cdots\cdots\cdots \qquad m_l = 0, 1, 2, \cdots\cdots$$

and the set of points $(t_{1i}, t_{2i}, \cdots\cdots, t_{li})$ corresponding to non-vanishing $\lambda_i$ forms the set $B_1$.

Again in order that a system of trials which is of the first class with respect to the first $l_1$ exclusive events, may be a determinate system with respect to a system of $l$ mutually exclusive events $(l > l_1)$ it is necessary and sufficient that

(i) the $l_1$ systems of simultaneous equations

$$T_{m_1, 0, \cdots 0} = \sum_{i=1}^{r_1} t_{1i}^{m_1} \lambda_{1i}, \qquad m_1 = 0, 1, 2, \cdots\cdots, 2r_1;$$

$$T_{0, m_2, 0 \cdots 0} = \sum_{i=1}^{r_2} t_{2i}^{m_2} \lambda_{2i}, \qquad m_2 = 0, 1, 2, \cdots\cdots, 2r_2;$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$T_{0, 0, \cdots m_{l_1}, 0, \cdots 0} = \sum_{i=1}^{r_{l_1}} t_{l_1 i}^{m_{l_1}} \lambda_{l_1 i}, \quad m_{l_1} = 0, 1, 2, \cdots\cdots, 2r_{l_1};$$

should have unique solutions respectively and also

(ii) $$T_{m_1, m_2, \cdots, m_l}$$

$$\left(\begin{array}{ll} m_1 = 0, 1, 2, \cdots\cdots, r_1 - 1; & m_{l_1+1} = 0, 1, 2, \cdots\cdots, \inf; \\ \cdots\cdots\cdots\cdots\cdots\cdots & \cdots\cdots \cdots \cdots \\ m_{l_1} = 0, 1, 2, \cdots\cdots, r_{l_1} - 1; & m_l = 0, 1, 2, \cdots\cdots, \inf. \end{array}\right)$$

should exist.

If these conditions be satisfied, then there exists a system of functions

$$\bar{\lambda}_{i_1, i_2 \cdots i_{l_1}}(t_{l_1+1}, t_{l_1+2} \cdots\cdots, t_l)$$

$$i_1 = 1, 2, 3, \cdots\cdots r_1\,;\ i_2 = 1, 2, \cdots\cdots, r_2\,;\ \cdots\cdots\cdots\cdots$$

of $l - l_1$ variables satisfying

$$T_{m_1 m_2 \cdots m_l} = \sum_{i_1=1}^{r_1} \cdots\cdots \sum_{i_{l_1}=1}^{r_{l_1}} t_{1 i_1}^{m_1} \cdots\cdots t_{l_1 i_{l_1}}^{m_{l_1}} \int t_{l_1+1}^{m_{l_1+1}} \cdots\cdots t_l^{m_l}\, d\bar{\lambda}_{i_1 i_2 \cdots i_{l_1}},$$

$$m_1 = 0, 1, 2, \cdots\cdots, r_1 - 1\,; \qquad m_{l_1+1} = 0, 1, 2, \cdots\cdots \text{inf.}\,;$$

$$\cdots\cdots\ \cdots\cdots \qquad \cdots\cdots\cdots\cdots$$

$$m_{l_1} = 0, 1, 2, \cdots\cdots, r_{l_1} - 1\,; \qquad m_l = 0, 1, 2, \cdots\cdots \text{inf.}\,.$$

where

$$\int t_{l_1+1}^{m_{l_1+1}} \cdots\cdots t_l^{m_l}\, d\bar{\lambda}_{i_1 i_2 \cdots i_{l_1}}$$

has the analogous meaning as defined by (201); and at points $(t_{l_1+1}, t_{l_1+2}, \cdots\cdots t_l)$ of continuity of these functions we have

$$\lim_{n=\infty} \sum F(n; s_1, s_2, \cdots\cdots, s_l) = \bar{\lambda}_{i_1 i_2 \cdots i_{l_1}}(t_{l_1+1}, t_{l_1+2}, \cdots\cdots, t_l),$$

the summation on the left being extended over

$$\left| \frac{s_1}{n} - t_{i_1} \right| \leqq \varepsilon, \quad \left| \frac{s_2}{n} - t_{i_2} \right| \leqq \varepsilon, \cdots\cdots \left| \frac{s_{l_1}}{n} - t_{l_1} \right| \leqq \varepsilon,$$

$$\frac{s_{l_1+1}}{n} \leqq t_{l_1+1}, \quad \frac{s_{l_1+2}}{n} \leqq t_{l_1+2}, \cdots\cdots \frac{s_l}{n} \leqq t_l\,.$$

**35.** We now give examples belonging to the class considered.

If, in the example considered in § 27, $P_m$, $Q_m$, $R_m$ and $P_m'$, $Q_m'$, $R_m'$ are subjected to the conditions

(a) $Q_m = R_m, \qquad P_m' = R_m',$

(b) $\lim_{m=\infty} m(1 - P_m) = k_1'$ and $\lim_{m=\infty} m R_m = k_1$

exist and none of them vanish,

(c) $Q_m' = q'$, a constant which is positive and smaller than 1,

then we get a system of games belonging to a determinate system of the generalized zero class with respect to the winning of $A$ and $B$. The

system of games is, however, of the first class with respect to *B's* winning.

Again if, in the above example, the condition ($c$) be replaced by that

$$(c') \qquad \lim_{m=\infty} m(1-Q_m')=k_2' \text{ and } \lim_{m=\infty} m R_m'=k_2$$

exist so that $k_2' \neq 0$, $k_2 \neq 0$ ([1]), then we get a determinate system of games of the generalized zero class with respect to the winning of $A$ and $B$ ([2]). The system of games is here also of the generalized zero class with respect to *A's* winning or with respect to *B's* winning.

We may construct further examples illustrating determinate systems of several classes. But as the calculation of $T_{m_1 m_2 \cdots}$ is, in many instances, complicated we shall not here enter into its discussions.

## Part VI.

### Further Extensions of the Results.

**36.** The results which have been obtained in the foregoing Parts may be looked upon as extensions of Poisson's Law of Large Numbers. We shall now go on to a short discussion of their generalization which corresponds to that considered by Tchebycheff.

Let $X_1, X_2, \cdots\cdots, X_n$ be $n$ variables which run over $n$ sets of values

$$\left.\begin{array}{l} a_{11}, a_{12}, \cdots\cdots, a_{1l_1}, \\ a_{21}, a_{22}, \cdots\cdots, a_{2l_2}, \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ a_{n1}, a_{n2}, \cdots\cdots, a_{nl_n}, \end{array}\right\} \qquad (202)$$

and the corresponding sets of their probabilities which may vary depending on the values taken by the other variables be

$$p_{11}, p_{12}, \cdots\cdots, p_{1l_1},$$
$$p_{21}, p_{22}, \cdots\cdots, p_{2l_2},$$

---

([1]) It is clear that $k_2' \geqq k_1$ and $k_1' \geqq k_2$ so that the two conditions ($b$) and ($c'$) may be replaced by $k_1 \neq 0$, $k_2 \neq 0$.

([2]) That all $T_{m_1 m_2}$, in this case, exist will readily be seen; but the actual calculation of them as well as the determination of the function $\pi(0, 0; x, y)$ is not easy unless certain conditions are fulfilled among $k_1, k_1', k_2, k_2'$.

$$\cdots\cdots\cdots$$

$$p_{n1}, p_{n2}, \cdots\cdots, p_{nl_n},$$

*à priori* such that

$$p_{i1}+p_{i2}+\cdots\cdots+p_{il_i}=1,$$

$$i=1, 2, \cdots\cdots.$$

Then a point $x$ is called the point at which the probability $\pi(x;\varepsilon)$ vanishes or the point of the class zero if, by taking a sufficiently small positive number $\varepsilon$, the probability that the inequality

$$\left|\frac{a_{1\alpha_1}+a_{2\alpha_2}+\cdots\cdots+a_{n\alpha_n}}{n}-x\right| \leqq \varepsilon,$$

i. e.

$$\left|\frac{\sum_1^n X_i}{n}-x\right| \leqq \varepsilon, \tag{203}$$

holds approaches zero as limit when $n$ tends to become infinite, and the set of such points is denoted by $A_0$. Similarly a point $x$ is called the point at which the probability $\pi(x;\varepsilon)$ is equal to $\lambda$ or the point of the first class, if the probability of the existence of (203) approaches a definite positive number $\lambda$ for all positive $\varepsilon$ smaller than a fixed positive number $\varepsilon_1$, and the set of such points is denoted by $A_1$.

Now confining ourselves to the *case where the double sequence* $(a_{ij})$ *is bounded* so that it has an upper limit $G$ and a lower limit $L$, it is easy to show as in the Part I that:

*In order that the set* $A_1$ *may consist of* $r$ *points, all other points in the interval* $(L, G)$ *belonging to the set* $A_0$ *it is necessary and sufficient* (1) *that the system of simultaneous equations*

$$T_m=\sum_{i=1}^r x_i^m \lambda_i,$$

$$m=0, 1, 2, \cdots\cdots, 2r,$$

*should have a unique solution, where* $T_m$ *is defined by*

$$T_m=\lim_{n=\infty} \sum_{\alpha_1=1}^{l_1} \sum_{\alpha_2=1}^{l_2} \cdots\cdots \sum_{\alpha_n=1}^{l_n} \left\{\frac{a_{1\alpha_1}+a_{2\alpha_2}+\cdots\cdots a_{n\alpha_n}}{n}\right\}^m (A_{\alpha_1} A_{\alpha_2} \cdots\cdots A_{\alpha_n})_p,$$

(1) This condition is still sufficient even if we remove the restriction that the double sequence $(a_{ij})$ is bounded as is easily seen from the discussion in § 3.

$(A_{\alpha_1} A_{\alpha_2} \cdots\cdots A_{\alpha_n})_p$ *being the probability of the concurrence of* $a_{1\alpha_1}, a_{2\alpha_2}, \cdots\cdots, a_{n\alpha_n}$ *of* $n$ *variables* $X_1, X_2, \cdots\cdots, X_n$.

Again if $\pi(L, x)$ denote the limit, when $n=\infty$, of the probability of the existence of

$$L \leqq \frac{\sum_1^n X_i}{n} \leqq x,$$

then we have, as in the Part III:

*In order that* $\pi(L, x)$ *exists at an everywhere dense set of points in the interval* $(L, G)$ *it is necessary and sufficient that all* $T_m$ $(m=0, 1, 2, \cdots\cdots)$ *should exist.*

*If all* $T_m$ *exist then the system of integral equations*

$$T_m = G^m - \int_L^G \frac{d}{dx}(x^m)\,\bar{\lambda}(x)\,dx,$$

$$m=1, 2, 3, \cdots\cdots (^1),$$

*is satisfied by a monotonic function* $\bar{\lambda}(x)$ *in the interval* $(L, G)$ *so that at points of its continuity we have*

$$\bar{\lambda}(x) = \pi(L, x).$$

It now remains to calculate $T_m$ in this case.

Let $(A_\alpha)_p$ denote the probability of the occurrence of $a_{1\alpha}$ of the variable $X_1$, $(EA_\alpha)_p$ that of $a_{2\alpha}$ of $X_2$; $(A_\alpha A_\beta)_p$ that of the concurrence of $a_{1\alpha}$ and $a_{2\beta}$ of two variables $X_1$ and $X_2$, and so on. Then as for such symbols the fundamental laws of algebra with the exception of the commutative law in the product expressions hold we have

$$\sum_{\alpha_1=1}^{l_1} \sum_{\alpha_2=1}^{l_2} \cdots\cdots \sum_{\alpha_n=1}^{l_n} a^m_{1\alpha_1} (A_{\alpha_1} A_{\alpha_2} \cdots\cdots A_{\alpha_n})_p$$

$$= \sum_{\alpha_1=1}^{l_1} a^m_{1\alpha_1} \sum_{\alpha_2=1}^{l_2} \cdots\cdots \sum_{\alpha_n=1}^{l_n} (A_{\alpha_1} A_{\alpha_2} \cdots\cdots A_{\alpha_n})_p$$

$$= \sum_{\alpha_1=1}^{l_1} a^m_{1\alpha_1} (A_{\alpha_1})_p$$

$$= \sum_{\alpha_1=1}^{l_1} a^m_{1\alpha_1} \cdot p_{1\alpha_1},$$

---

(1) The relation corresponding to $m=0$, is to be replaced by

$$T_0 = 1.$$

$$\sum_{\alpha_1=1}^{l_1}\sum_{\alpha_2=1}^{l_2}\cdots\cdots\sum_{\alpha_n=1}^{l_n} a^{m}_{2\alpha_2}(A_{\alpha_1}A_{\alpha_2}\cdots\cdots A_{\alpha_n})_p$$

$$=\sum_{\alpha_2=1}^{l_2} a^{m}_{2\alpha_2}\cdot\sum_{\alpha_1=1}^{l_1}\sum_{\alpha_3=1}^{l_3}\cdots\cdots\sum_{\alpha_n=1}^{l_n}(A_{\alpha_1}A_{\alpha_2}\cdots\cdots A_{\alpha_n})_p$$

$$=\sum_{\alpha_2=1}^{l_2} a^{m}_{2\alpha_2}(EA_{\alpha_2})_p$$

$$=\sum_{\alpha_2=1}^{l_2} a^{m}_{2\alpha_2}p_{2\alpha_2},$$

.....

so that

$$\sum_{\alpha_1=1}^{l_1}\sum_{\alpha_2=1}^{l_2}\cdots\cdots\sum_{\alpha_n=1}^{l_n}\left\{a^{m}_{1\alpha_1}+a^{m}_{2\alpha_2}+\cdots\cdots+a^{m}_{n\alpha_n}\right\}(A_{\alpha_1}A_{\alpha_2}\cdots\cdots A_{\alpha_n})_p$$

$$=\sum_{i=1}^{n}\sum_{\alpha_i=1}^{l_i} a^{m}_{i\alpha_i}p_{i\alpha_i}. \tag{204}$$

Similarly

$$\sum_{\alpha_1=1}^{l_1}\cdots\cdots\sum_{\alpha_n=1}^{l_n}\left\{a^{m-1}_{1\alpha_1}a_{2\alpha_2}+\cdots\cdots+a_{n-1\,\alpha_{n-1}}a^{m-1}_{n\alpha_n}\right\}(A_{\alpha_1}A_{\alpha_2}\cdots\cdots A_{\alpha_n})_p$$

$$=\sum_{\alpha_1=1}^{l_1}\sum_{\alpha_2=1}^{l_2} a^{m-1}_{1\alpha_1}a_{2\alpha_2}(A_{\alpha_1}A_{\alpha_2})_p+\cdots\cdots$$

$$+\sum_{\alpha_{n-1}=1}^{l_{n-1}}\sum_{\alpha_n=1}^{l_n} a_{n-1\,\alpha_{n-1}}a^{m-1}_{n\alpha_n}(E^{n-2}A_{\alpha_{n-1}}A_{\alpha_n})_p, \tag{205}$$

.....

$$\sum_{\alpha_1=1}^{l_1}\cdots\cdots\sum_{\alpha_n=1}^{l_n}\left\{a_{1\alpha_1}\cdots\cdots a_{m\alpha_m}+\cdots\cdots+a_{n-m+1\,\alpha_{n-m+1}}\cdots\cdots a_{n\alpha_n}\right\}(A_{\alpha_1}A_{\alpha_2}\cdots\cdots A_{\alpha_n})_p$$

$$=\left\{\sum_{\alpha_1=1}^{l_1}\cdots\cdots\sum_{\alpha_m=1}^{l_m} a_{1\alpha_1}\cdots\cdots a_{m\,\alpha_m}(A_{\alpha_1}A_{\alpha_2}\cdots\cdots A_{\alpha_m})_p+\cdots\cdots\right.$$

$$\left.+\sum_{\alpha_{n-m+1}=1}^{l_{n-m+1}}\cdots\cdots\sum_{\alpha_n=1}^{l_n} a_{n-m+1\,\alpha_{n-m+1}}\cdots\cdots a_{n\alpha_n}(E^{n-m}A_{\alpha_{n-m+1}}\cdots\cdots A_{\alpha_n})_p\right\}. \tag{206}$$

Consequently we get

$$\sum_{\alpha_1=1}^{l_1}\cdots\cdots\sum_{\alpha_n=1}^{l_n}\left\{a_{1\alpha_1}+a_{2\alpha_2}+\cdots\cdots+a_{n\alpha_n}\right\}^m(A_{\alpha_1}A_{\alpha_2}\cdots\cdots A_{\alpha_n})_p$$

$$=A_m(n)+\frac{m!}{(m-1)!}B_m(n)+\cdots\cdots+m!\ G_m(n),$$

where $A_m(n)$, $B_m(n)$, $\cdots\cdots$, $G_m(n)$ denote the expressions which stand on the left of (204), (205), $\cdots\cdots$, (206) respectively.

If now the double sequence $(a_{ij})$ be bounded, it will easily be seen that $A_m(n)$, $B_m(n)$, $\cdots\cdots$ except $G_m(n)$ are all of lower order than $n^m$ when $n$ becomes infinite and hence we have

$$T_m=\lim_{n=\infty}\sum_{\alpha_1=1}^{l_1}\cdots\cdots\sum_{\alpha_n=1}^{l_n}\left\{\frac{a_{1\alpha_1}+\cdots\cdots+a_{n\alpha_n}}{n}\right\}^m(A_{\alpha_1}A_{\alpha_2}\cdots\cdots A_{\alpha_n})_p$$

$$=m!\lim_{n=\infty}\frac{G_m(n)}{n^m}, \tag{207}$$

*a generalized formula of* (37).

It is evident that several results which correspond to those obtained in the Parts I, II and III may be derived from the criterion given in the present article.

**37.** Again let $X_i'$ and $X_i''$ be two exclusive complementary variables which run over two sets of values

$$a_{i1},\ a_{i2},\ \cdots\cdots,\ a_{iu'}\ \text{and}\ a_{i(u'+1)},\ \cdots\cdots,\ a_{iu},$$

respectively, i.e. when $X_i'$ takes any one of the former values, $X_i''$ takes no value while when $X_i''$ takes any one of the latter values, $X_1'$ takes no value and also there exists

$$p_{i1}+p_{i2}+\cdots\cdots+p_{iu'}+\cdots\cdots+p_{iu}=1.$$

Then the set of points in two dimensional domain such that, by taking a sufficiently small positive number $\varepsilon$, the probability that the inequalities

$$\left|\frac{\sum_1^n X'}{n}-x\right|\leqq\varepsilon,\quad\left|\frac{\sum_1^n X_i''}{n}-y\right|\leqq\varepsilon,$$

hold simultaneously approaches zero as limit when $n$ becomes infinite, is denoted by $A_0$. Similarly for the set $A_1$.

Now confining ourselves, as in the above, to the *case where the*

*double sequence* $(a_{ij})$ *is bounded* we have the result which corresponds to that in the Part IV:

*In order that there exists no point in the two dimensional domain*

$$L \leqq x, \qquad L \leqq y, \qquad x+y \leqq G,$$

*which belongs neither to the set* $A_0$ *nor to the set* $A_1$ *it is necessary and sufficient that there should exist two positive integers* $r_1$ *and* $r_2$ *such that the two systems of simultaneous equations*

$$T_{m,0} = \sum_{i=1}^{r_1} x_i^m \lambda_i, \qquad m=0, 1, 2, \cdots\cdots, 2r_1;$$

*and*

$$T_{0,m} = \sum_{i=1}^{r_2} x_i^m \lambda_i', \qquad m=0, 1, 2, \cdots\cdots, 2r_2,$$

*will have unique solutions respectively and also*

$$T_{m_1, m_2} \begin{pmatrix} m_1=0, 1, 2, \cdots\cdots, r_1-1 \\ m_2=0, 1, 2, \cdots\cdots, r_2-1 \end{pmatrix}$$

*should exist, where*

$$T_{m_1 m_2} = \lim_{n=\infty} T_{m_1, m_2}(n)$$

$$T_{m_1 m_2}(n) = \sum_{\alpha_1=1}^{l_1} \cdots\cdots \sum_{\alpha_n=1}^{l_n} \left\{ \frac{a'_{1\alpha_1} + \cdots\cdots + a'_{n\alpha_n}}{n} \right\}^{m_1}$$

$$\times \left\{ \frac{a''_{1\alpha_1} + \cdots\cdots + a''_{n\alpha_n}}{n} \right\}^{m_2} (A_{\alpha_1} A_{\alpha_2} \cdots\cdots A_{\alpha_n})_{pr}$$

$a'_{i\alpha_i}$ *and* $a''_{i\alpha_i}$ *being defined by*

$$\begin{aligned} a'_{i\alpha_i} &= a_{i\alpha_i} && \text{if } \alpha_i = 1, 2, \cdots\cdots, l_i', \\ &= 0 && \text{if } \alpha_i = l_i'+1, \cdots\cdots, l_i; \\ a''_{i\alpha_i} &= 0 && \text{if } \alpha_i = 1, 2, \cdots\cdots, l_i', \\ &= a_{i\alpha_i} && \text{if } \alpha_i = l_i'+1, \cdots\cdots, l_i. \end{aligned}$$

*Furthermore we may arrive at a generalized result which corresponds to that obtained in the Part V.*

A formula for the calculation of $T_{m_1, m_2}$ may be derived in a quite similar manner as in the last article and we have

$$T_{m_1 m_2} = m_1!\ m_2!\ \lim_{n=\infty} \frac{G_{m_1 m_2}(n)}{n^{m_1+m_2}},$$

where

$$G_{m_1 m_2}(n) = \Sigma\left\{(a_{1\alpha_1}\cdots\cdots a_{m_1\alpha_{m_1}})(a_{m_1+1\ \alpha_{m_1-1}}\cdots\cdots a_{m_1+m_2\ \alpha_{m_1+m_2}})\right\}$$

$$\times(A_{\alpha_1} A_{\alpha_2}\cdots\cdots A_{\alpha_{m_1+m_2}})_p + \cdots\cdots$$

$$+\Sigma'\left\{(a_{n-m_1-m_2+1\ \alpha_{n-m_1-m_2+1}}\cdots\cdots a_{n-m_1\ \alpha_{n-m_1}})\right.$$

$$\left.\times(a_{n-m_1+1\ \alpha_{n-m_1+1}}\cdots\cdots a_{n\alpha_n})\right\}(E^{n-m_1-m_2} A_{\alpha_{n-m_1-m_2+1}}\cdots\cdots A_{\alpha_n})_p,$$

the first summation $\Sigma$ and the last summation $\Sigma'$ being extended over

$$\alpha_1 = 1, 2, \cdots\cdots, l_1';$$

$$\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\alpha_{m_1} = 1, 2, \cdots\cdots, l'_{m_1};$$

$$\alpha_{m_1+1} = l'_{m_1+1}+1, \cdots\cdots, l_{m_1+1};$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\alpha_{m_1+m_2} = l'_{m_1+m_2}+1, \cdots\cdots, l_{m_1+m_2},$$

and

$$\alpha_{n-m_1-m_2+1} = l'_{n-m_1-m_2+1}+1, \cdots\cdots, l_{n-m_1-m_2+1};$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\alpha_{n-m_1} = l'_{n-m_1}+1, \cdots\cdots, l_{n-m_1};$$

$$\alpha_{n-m_1+1} = 1, 2, 3, \cdots\cdots, l'_{n-m_1+1};$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\alpha_n = 1, 2, 3, \cdots\cdots, l'_n,$$

respectively, all other summations being similarly performed between proper limits. This is a generalized formula of (150) for $l=2$.

Thus a generalization of the results obtained in the Parts IV and V has been established for $l=2$. It is evident that the similar result holds for any $l$.

**38.** In the last two articles we have confined ourselves to the cases where $l_1, l_2, \cdots\cdots, l_n$ are all finite. If some or all of them become in-

finite, we have to deal with infinite series, instead of the finite expressions, which will however be all absolutely convergent in virtue of the relation

$$\sum_{\alpha_1=1}^{\infty} \sum_{\alpha_2=1}^{\infty} \cdots\cdots \sum_{\alpha_n=1}^{\infty} (A_{\alpha_1} A_{\alpha_2} \cdots\cdots A_{\alpha_n})_p = 1,$$

if we assume, as in the above, that the double sequence $(a_{ij})$ is bounded.

It then follows that *results which have been obtained in the last two articles equally hold even for the case when the restrictions that* $l_1, l_2, \cdots\cdots \cdots\cdots, l_n$ *are all finite, are removed*(1).

Also the reasoning given in the last articles will equally be applied to the cases where the variables $X_1, X_2, \cdots\cdots, X_n$ run over, instead of enumerable sets of values, all values within the intervals

$$(\beta_1, \gamma_1), \quad (\beta_2, \gamma_2), \cdots\cdots, \quad (\beta_n \gamma_n),$$

respectively, of which the sequences $(\beta_n)$ and $(\gamma_n)$ are both bounded; and we have the same theorem, corresponding to that obtained in § 36, by merely replacing the expression of $T_m$ by

$$T_m = \lim_{n=\infty} \int_{\beta_1}^{\gamma_1} \cdots\cdots \int_{\beta_n}^{\gamma_n} \left\{ \frac{X_1 + X_2 + \cdots\cdots + X_n}{n} \right\}^m$$
$$\times W(X_1, X_2, \cdots\cdots, X_n)\, dX_1\, dX_2 \cdots\cdots dX_n,$$

i.e.

$$T_m = m!\, \lim_{n=\infty} \left\{ \int_{\beta_1}^{\gamma_1} \cdots\cdots \int_{\beta_m}^{\gamma_m} X_1 \cdots\cdots X_m \,.\, W(X_1, \cdots\cdots, X_m)\, dX_1 \cdots\cdots dX_m \right.$$
$$+ \cdots\cdots\cdots$$
$$\left. + \int_{\beta_{n-m+1}}^{\gamma_{n-m+1}} \cdots\cdots \int_{\beta_n}^{\gamma_n} X_{n-m+1} \cdots\cdots X_n\, W(X_{n-m+1}, \cdots\cdots, X_n)\, dX_{n-m+1} \cdots\cdots dX_n \right\},$$

where $W(X_1, X_2, \cdots\cdots, X_n)$ is the probability function of $n$ variables $X_1, X_2, \cdots\cdots, X_n$, i.e. the probability of the concurrence of them, $W(X_1, X_2, \cdots\cdots, X_m)$ that of the first $m$ variables $X_1, X_2, \cdots\cdots, X_m$ and so on,

(1) The set of values taken by the variable $X$ may of course not be discrete; for example the set may consist of all rational numbers between 0 and 1.

$W(X_{n-m+1}, \cdots\cdots, X_n)$ that of the last $m$ variables $X_{n-m+1}, \cdots\cdots, X_n$; and similarly by dividing each of the intervals above considered into two or more parts we may derive another theorem which corresponds to that in § 37.

## APPENDIX.

### Note on a Simple Determinate System of Non-independent Trials of the First Class.

If $F(n, s)$ denote the probability of the happening just $s$ times of an event in $n$ trials, and the limit to which the sum, with respect to $s$,

$$\varphi_n(x) = \sum_{0 \leqq \frac{s}{n} \leqq x} F(n, s) \tag{1}$$

tends when $n$ becomes infinite, be denoted by $\bar{\lambda}(x)$, then a system of non-independent trials for which the function $\bar{\lambda}(x)$ becomes discontinuous at a finite number of points in the interval $(0 \leqq x \leqq 1)$ and is constant within each interval of its continuity has been termed a simple determinate system of the first class with respect to the event and the points of discontinuity of the function $\bar{\lambda}(x)$ have been called points of the first class. I have shown in the Part I([1]) that, putting

$$T_m = \lim_{n=\infty} \sum_{s=0}^{n} \left( \frac{s}{n} \right)^m F(n, s), \tag{2}$$

*the condition necessary and sufficient in order that a system of non-independent trials may be a simple determinate system of the first class with respect to the event under consideration having $r$ points of the first class is that*

(i) *the simultaneous equations*

$$T_m = \sum_{i=1}^{r} x_i^m \lambda_i \text{([2])},$$

$$m = 0, 1, 2, \cdots\cdots, 2r,$$

*should have a unique solution $(x_i, \lambda_i)$ or*

---

([1]) § 5 and § 6.

([2]) Of course the existence of $T_m$ is necessary.

(ii)
$$D_r = \begin{vmatrix} 1 & T_1 & T_2 & \cdots\cdots & T_r \\ T_1 & T_2 & T_3 & \cdots\cdots & T_{r+1} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ T_r & T_{r+1} & T_{r+2} & \cdots\cdots & T_{2r} \end{vmatrix} = 0, \tag{3}$$

$$D_{r-1} = \begin{vmatrix} 1 & T_1 & \cdots\cdots & T_{r-1} \\ T_1 & T_2 & \cdots\cdots & T_r \\ \cdots & \cdots & \cdots & \cdots \\ T_{r-1} & T_r & \cdots\cdots & T_{2r-2} \end{vmatrix} \neq 0, \tag{4}$$

*and the equation*

$$H_r(x) = \begin{vmatrix} 1 & x & x^2 & \cdots\cdots & x^r \\ 1 & T_1 & T_2 & \cdots\cdots & T_r \\ T_1 & T_2 & T_3 & \cdots\cdots & T_{r+1} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ T_{r+1} & T_r & T_{r+1} & \cdots\cdots & T_{2r-1} \end{vmatrix} = 0 \tag{5}$$

*should have no multiple roots*( [1] ).

It seems, however, probable that the condition (5) is superfluous( [2] ), i.e. the two relations $D_r = 0$ and $D_{r-1} \neq 0$ are sufficient to ensure the existence of a simple determinate system of the first class. That this is really the case will be shown in the article 1 of the present note.

Again the criterion (i) shows us that a simple determinate system of the first class with respect to an event is characterized by some relations among $T_1, T_2, \cdots\cdots, T_{2r}$. It is expected and can be shown that such a system of trials is also characterized by analogous relations among any $2r$ ones of $T_1, T_2, \cdots\cdots, T_\rho, \cdots\cdots$. The articles 2 and 3 of the present note will be devoted to the deduction of those relations.

**1.** In order to show that the condition (5) is superfluous we establish the following lemma.

*Lemma. If*

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( [1] ) The symbols $D_{r-1}$ and $H_r(x)$ here used is connected with the symbols $\Delta_r$ and $\Delta_r(x)$ in § 6, by $D_{r-1} = \frac{1}{2}(r-1)\, r\, \Delta_r$, $H_r(x) = \frac{1}{2}\, r(r+1) \cdot \Delta_r(x)$.

( [2] ) In the sense that this condition is a direct consequence of (4).

$$D_0=1, \quad D_1=\begin{vmatrix} 1 & T_1 \\ T_1 & T_2 \end{vmatrix} \neq 0, \cdots\cdots D_{r-1}=\begin{vmatrix} 1 & T_1 & T_2 & \cdots\cdots & T_{r-1} \\ T_1 & T_2 & T_3 & \cdots\cdots & T_r \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ T_{r-1} & T_r & T_{r+1} & \cdots\cdots & T_{2r-2} \end{vmatrix} \neq 0,$$

*where $T$ is defined by* (2), *then*

$$D_\rho > 0, \qquad (\rho=0, 1, 2, \cdots\cdots, r-1)$$

*and the equation*

$$H_r(x)=\begin{vmatrix} 1 & x & x^2 & \cdots\cdots\cdots & x^r \\ 1 & T_1 & T_2 & \cdots\cdots\cdots & T_r \\ T_1 & T_2 & T_3 & \cdots\cdots\cdots & T_{r+1} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ T_{r-1} & T_r & T_{r+1} & \cdots\cdots\cdots & T_{2r-1} \end{vmatrix}=0$$

*will have no multiple roots*(1).

From the expression of $D_\rho$ we have

$$D_\rho = \lim_{n=\infty} D_\rho(n),$$

where

$$D_\rho(n)=\begin{vmatrix} \Sigma F(n, s), & \Sigma \frac{s}{n} F(n, s), & \cdots\cdots\cdots, & \Sigma \left(\frac{s}{n}\right)^\rho F(n, s) \\ \Sigma \frac{s}{n} F(n, s), & \Sigma \left(\frac{s}{n}\right)^2 F(n, s), & \cdots\cdots, & \Sigma \left(\frac{s}{n}\right)^{\rho+1} F(n, s) \\ \cdots & \cdots & \cdots & \cdots \\ \Sigma \left(\frac{s}{n}\right)^\rho F(n, s), & \Sigma \left(\frac{s}{n}\right)^{\rho+1} F(n, s), & \cdots\cdots, & \Sigma \left(\frac{s}{n}\right)^{2\rho} F(n, s) \end{vmatrix},$$

$\Sigma$ denoting the summation with respect to $s$ from 0 to $n$.

Now the matrix

(1) This elementary property of an algebraic equation as well as that in § 2 is probably well known. But as I could not find them out in any text book within my reach I shall here give a proof of them.

$$\begin{pmatrix} \Sigma F(n,s) & \cdots\cdots\cdots & \Sigma\left(\frac{s}{n}\right)^{\rho} F(n,s) \\ \cdots & \cdots & \cdots \\ \Sigma\left(\frac{s}{n}\right)^{\rho} F(n,s) & \cdots\cdots & \Sigma\left(\frac{s}{n}\right)^{2\rho} F(n,s) \end{pmatrix}$$

can be composed from two matrices

$$\begin{pmatrix} 1 & 1 & 1 & \cdots\cdots & 1 \\ 0 & \frac{1}{n} & \frac{2}{n} & \cdots\cdots & \frac{n}{n} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \left(\frac{1}{n}\right)^{\rho} & \left(\frac{2}{n}\right)^{\rho} & \cdots\cdots & \left(\frac{n}{n}\right)^{\rho} \end{pmatrix}$$

and

$$\begin{pmatrix} F(n,0) & 0 & \cdots\cdots & 0 \\ F(n,1) & \frac{1}{n} F(n,1) & \cdots\cdots & \left(\frac{1}{n}\right)^{\rho} F(n,1) \\ \cdots & \cdots & \cdots & \cdots \\ F(n,n) & \frac{n}{n} F(n,n) & \cdots\cdots & \left(\frac{n}{n}\right)^{\rho} F(n,n) \end{pmatrix},$$

so that

$$D_{\rho}(n) = \Sigma F(n,i_0)\, F(n,i_1) \cdots\cdots F(n,i_{\rho}) \begin{vmatrix} 1 & 1 & \cdots\cdots & 1 \\ \frac{i_0}{n} & \frac{i_1}{n} & \cdots\cdots & \frac{i_{\rho}}{n} \\ \cdots & \cdots & \cdots & \cdots \\ \left(\frac{i_0}{n}\right)^{\rho} & \left(\frac{i_1}{n}\right)^{\rho} & \cdots\cdots & \left(\frac{i_{\rho}}{n}\right)^{\rho} \end{vmatrix}^2,$$

where $\Sigma$ denotes the summation about $i_0, i_1, \cdots\cdots, i_{\rho}$ for all possible sets of integers satisfying

$$0 \leqq i_0 < i_1 < i_2 < \cdots\cdots < i_{\rho} \leqq n.$$

Consequently we get

$$D_{\rho}(n) \geqq 0,$$

and therefore

$$D_\rho \geqq 0,$$

which, combined with the assumption $D_\rho \neq 0$, gives

$$D_\rho > 0.$$

This proves the first part of the lemma.

Again expanding the determinant

$$H_\rho(x) = \begin{vmatrix} 1 & x & x^2 & \cdots\cdots x^\rho \\ 1 & T_1 & T_2 & \cdots\cdots T_\rho \\ \cdots & \cdots & \cdots & \cdots\cdots \\ T_{\rho-1} & T_\rho & T_{\rho+1} & \cdots\cdots T_{2\rho-1} \end{vmatrix},$$

in terms of the constituents of the first row we obtain

$$H_\rho(x) = A_0 + A_1 x + A_2 x^2 + \cdots\cdots + A_\rho x^\rho,$$

where the constants $A_0, A_1, \cdots\cdots, A_\rho$ satisfy

$$\left.\begin{aligned} & A_0 + A_1 T_1 + A_2 T_2 + \cdots\cdots + A_\rho T_\rho = 0, \\ & A_0 T_1 + A_1 T_2 + A_2 T_3 + \cdots\cdots + A_\rho T_{\rho+1} = 0, \\ & \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ & A_0 T_{\rho-1} + A_1 T_\rho + A_2 T_{\rho+1} + \cdots\cdots + A_\rho T_{2\rho-1} = 0, \\ & A_0 T_\rho + A_1 T_{\rho+1} + A_2 T_{\rho+2} + \cdots\cdots + A_\rho T_{2\rho} = (-1)^\rho D_\rho, \end{aligned}\right\} \quad (6)$$

and

$$A_\rho = (-1)^\rho D_{\rho-1}. \quad (7)$$

Consequently by adding to the last column of the determinant

$$A_\rho H_{\rho+1}(x) = \begin{vmatrix} 1, & x, & \cdots\cdots, & x^\rho, & A_\rho x^{\rho+1} \\ 1, & T_1, & \cdots\cdots, & T_\rho, & A_\rho T_{\rho+1} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ T_\rho, & T_{\rho+1}, & \cdots\cdots, & T_{2\rho}, & A_\rho T_{2\rho+1} \end{vmatrix}$$

the second column multiplied by $A_0$, the third by $A_1, \cdots\cdots$, the $(\rho+1)$th by $A_{\rho-1}$ we get

$$A_\rho H_{\rho+1}(x)=\begin{vmatrix} 1, & x, & \cdots\cdots, & x^\rho, & x\,H_\rho(x) \\ 1, & T_1, & \cdots\cdots, & T_\rho, & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ T_{\rho-2}, & T_{\rho-1}, & \cdots\cdots, & T_{2\rho-2}, & 0 \\ T_{\rho-1}, & T_\rho, & \cdots\cdots, & T_{2\rho-1}, & (-1)^\rho D_\rho \\ T_\rho, & T_{\rho+1}, & \cdots\cdots, & T_{2\rho}, & L \end{vmatrix},$$

where

$$L=A_0\,T_{\rho+1}+A_1\,T_{\rho+2}+\cdots\cdots+A_\rho\,T_{2\rho+1},$$

and again by adding to the last but one column of the determinant

$$A_\rho^2\,H_{\rho+1}(x)=\begin{vmatrix} 1, & x, & \cdots\cdots, & A_\rho\,x^\rho, & x\,H_\rho(x) \\ 1, & T_1, & \cdots\cdots, & A_\rho\,T_\rho, & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ T_{\rho-1}, & T_\rho, & \cdots\cdots, & A_\rho\,T_{2\rho-1}, & (-1)^\rho D_\rho \\ T_\rho, & T_{\rho+1}, & \cdots\cdots, & A_\rho\,T_{2\rho}, & L \end{vmatrix}$$

the first column multiplied by $A_0$, the second by $A_1, \cdots\cdots$, the $\rho$ th by $A_{\rho-1}$ we have

$$A_\rho^2 H_{\rho+1}(x)=\begin{vmatrix} 1, & x, & \cdots\cdots, & x^{\rho-1}, & H_\rho(x), & x\,H_\rho(x) \\ 1, & T_1, & \cdots\cdots, & T_{\rho-1}, & 0, & 0, \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ T_{\rho-1}, & T_\rho, & \cdots\cdots, & T_{2\rho-2}, & 0, & (-1)^\rho D_\rho \\ T_\rho, & T_{\rho+1}, & \cdots\cdots, & T_{2\rho-1}, & (-1)^\rho D_\rho, & L \end{vmatrix},$$

from which we obtain

$$A_\rho^2\,H_{\rho+1}(x)=(-D_{\rho-1}\,.\,D_\rho\,.\,x+K)\,H_\rho(x)-D_\rho^2\,H_{\rho-1}(x),$$

where $K$ is a constant independent of $x$, or, by the relation (7),

$$D_{\rho-1}^2\,H_{\rho+1}(x)=(-D_{\rho-1}\,D_\rho\,x+K)\,H_\rho(x)-D_\rho^2\,H_{\rho-1}(x). \qquad (8)$$

This formula holds even for the case when $\rho=1$ if we assume that

$$H_0(x)=1.$$

Now in virtue of the supposition

$$D_\rho \neq 0, \qquad (\rho=0, 1, 2, \cdots\cdots, r-1)$$

$H_\rho(x)$ will be a polynomial of $x$ of the degree $\rho$ $(\rho=1, 2, \cdots\cdots, r)$ and by aid of the formula just obtained we may easily conclude as follows.

The series of the functions

$$H_r(x), \quad H_{r-1}(x), \cdots\cdots, \quad H_1(x), \quad H_0(x). \tag{9}$$

satisfies the following conditions:—

(*a*) the last function $H_0(x)$ is positive for all values of $x$,

(*b*) any two consecutive functions do not vanish for the same of $x$,

(*c*) the two functions $H_{\rho+1}(x)$ and $H_{\rho-1}(x)$ have the contrary signs for any real root of $H_\rho(x)=0$.

Then it results by Sturm's theorem that if $N$ be the number of real and distinct roots of $H_r(x)=0$, and $V(\infty)$ and $V(-\infty)$ denote the numbers of variations of signs in the series (9) when $+\infty$ and $-\infty$ are substituted for $x$ respectively then

$$r \geqq N \geqq | V(\infty) - V(-\infty) |.$$

Since the leading coefficient of $H_\rho(x)$ is

$$(-1)^\rho D_\rho,$$

we have evidently

$$V(\infty)=r, \qquad V(-\infty)=0,$$

and therefore

$$| V(\infty) - V(-\infty) | = r.$$

It follows that

$$N=r,$$

so that the equation

$$H_r(x)=0$$

will have all its roots real and distinct.

This proves the second part of the lemma.

By virtue of this lemma and by aid of the criterion (ii) we are now led to the conclusion:

The conditions

$$D_\rho \neq 0 \ (\rho=0, 1, 2, \cdots\cdots r-1) \quad \text{and} \quad D_r=0$$

are sufficient to ensure the occurrence of a simple determinate system of the first class with respect to an event having $r$ points of the first class.

And since, in this case, we have also

$$D_{r+1}=D_{r+2}=\cdots\cdots=0\text{(}^{1}\text{)}, \tag{10}$$

we may derive the following criterion:—

*In order that a system of trials may be a simple determinate system of the first class having $r$ points of the first class it is necessary and sufficient that*

$$D_{r-1}\neq 0 \text{ and } D_r=0.$$

That the condition is necessary is evident. That the condition is sufficient can be shown as follows.

If $D_{r-1}\neq 0$ and $D_r=0$ then there will exist an integer $r_1$ $(r_1\leqq r)$ such that

$$D_\rho\neq 0 \ (\rho=0, 1, \cdots\cdots, r_1-1) \text{ and } D_{r_1}=0,$$

so that the system of trials belongs to the first class having $r_1$ points of the first class. We have therefore, if $r_1<r$, by (10)

$$D_{r-1}=0,$$

which contradicts the assumption and hence

$$r_1=r.$$

**2.** We now go on to the derivation of conditions which are to be satisfied by

$$T_{m_1},\ T_{m_2}, \cdots\cdots,\ T_{m_{2r}},$$

in order that a system of trials belongs to the first class.

It is of course evident that there should exist, for a properly chosen integer $r$, a solution $(x_i, \lambda_i)$ satisfying

$$T_m=\sum_{i=1}^{r} x_i^m \lambda_i, \tag{11}$$

$$m=0,\ m_1, m_2, \cdots\cdots, m_{2r},$$

with the conditions

$$0\leqq x_i\neq x_k\leqq 1, \qquad \lambda_i\neq 0, \tag{12}$$

if a system of trials be of the first class.

---

(1) This follows from the fact that the simultaneous equations

$$T_m=\sum_{i=1}^{k} x_i^m \lambda_i \qquad (m=0, 1, 2, \cdots, 2r, 2r+1, \cdots, 2k)$$

are always consistent.

To show that the converse is also true, we make use of a lemma relating to an algebraic equation which runs as follows:—

*Lemma. If*

$$f(x)=a_0+a_1x+a_2x^2+\cdots\cdots+a_kx^k=0 \qquad (a_k\neq 0)$$

*be a given equation of the $k^{th}$ degree having all its root positive*(1), *and $m_0, m_1, \cdots\cdots, m_k$ be a given series of $k+1$ integers such that*

$$0\leqq m_0<m_1<m_2<\cdots\cdots<m_k,$$

*then we may always construct an equation, with real coefficients, of the form*

$$\psi(x)=A_0x^{m_0}+A_1x^{m_1}+A_2x^{m_2}+\cdots\cdots+A_kx^{m_k}=0, \quad (A_k\neq 0)$$

*such that it has the same positive roots as the equation $f(x)=0$.*

Taking an integer $m$, let us consider a system of $m+1$ equations

$$\left.\begin{array}{c} f(x)=a_0+a_1x+a_2x^2+\cdots\cdots+a_kx^k=0, \\ xf(x)=a_0x+a_1x^2+a_2x^3+\cdots\cdots+a_kx^{k+1}=0, \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ x^mf(x)=a_0x^m+a_1x^{m+1}+\cdots\cdots+a_kx^{k+m}=0, \end{array}\right\} \qquad (13)$$

whose coefficients of $x^0, x, x^2, \cdots\cdots x^{k+m}$ form the $(m+1)-$rowed and $(k+m+1)-$columned matrix $M_m$ where

$$M_m=\left(\begin{array}{l} a_0,\ a_1,\ a_2,\ \cdots\cdots,\ a_k,\ \ 0,\ \ 0,\ \cdots\cdots,\ 0 \\ 0,\ \ a_0,\ a_1,\ \cdots\cdots,\ a_{k-1},\ a_k,\ 0,\ \cdots\cdots,\ 0 \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \end{array}\right),$$

or putting

$$a_{-1}=a_{-2}=\cdots\cdots=0,$$

$$a_{k+1}=a_{k+2}=\cdots\cdots=0,$$

$$M_m=\left(\begin{array}{l} a_0,\ \ a_1,\ a_2,\ \cdots\cdots,\ a_{k+m} \\ a_{-1},\ a_0,\ a_1,\ \cdots\cdots,\ a_{k+m-1} \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ a_{-m},\ a_{-m+1},\cdots\cdots,\ a_k \end{array}\right).$$

It can be shown that *no determinant of order $m+1$ formed from the matrix $M_m$ vanishes if the equation $f(x)=0$ have all its roots positive.*

That this holds when $m=0$ can at once be seen, since none of $a_0$,

(1) Some of them may of course be equal to one another. In this case they are to be counted as many times as the degree of their multiplicity.

$a_1, a_2, \cdots\cdots, a_k$ vanishes if $f(x)=0$ have all its roots positive. Now assume that this holds for a certain value of $m$ but not for $m$ increased by 1, e.g.,

$$\begin{vmatrix} a_{i_2-1} & a_{i_3-1} \cdots\cdots & a_{i_{m+2}-1} \\ a_{i_2-2} & a_{i_3-2} \cdots\cdots & a_{i_{m+2}-2} \\ \cdots\cdots & \cdots\cdots & \cdots\cdots \\ a_{i_2-m-1} & \cdots\cdots\cdots & a_{i_{m+2}-m-1} \end{vmatrix} \neq 0, \tag{14}$$

and

$$\Delta = \begin{vmatrix} a_{i_1} & a_{i_2} & \cdots\cdots & a_{i_{m+2}} \\ a_{i_1-1} & a_{i_2-1} & \cdots\cdots & a_{i_{m+2}-1} \\ \cdots\cdots & \cdots\cdots & \cdots\cdots & \cdots\cdots \\ a_{i_1-m-1} & \cdots\cdots & \cdots\cdots & a_{i_{m+2}-m-1} \end{vmatrix} = 0. \tag{15}$$

Then denoting the cofactors of the constituents of the first column in $\Delta$ by

$$\Delta_1, \Delta_2, \cdots\cdots, \Delta_{m+2},$$

the first of which at least does not vanish by the assumption, we have from (13)

$$(\Delta_1 + \Delta_2 x + \cdots\cdots + \Delta_{m+2} x^{m+1}) f(x)$$
$$= B_0 + B_1 x + \cdots\cdots + B_{k+m+1} x^{k+m+1}, \tag{16}$$

where

$$B_{i_2} = B_{i_3} = \cdots\cdots = B_{i_{m+2}} = 0, \tag{17}$$

and also, by the assumption (15),

$$B_{i_1} = 0. \tag{18}$$

The left hand side of identity (16) does not vanish identically on account of the relation $\Delta_1 \neq 0$, so that

$$B_0 + B_1 x + \cdots\cdots + B_{k+m+1} x^{k+m+1} = 0 \tag{19}$$

will be an equation but not an identity. Then it follows that the equation (19) will have at least $k$ positive roots which are the roots of the equation $f(x)=0$. But this can not be the case; for since the equation (19) has, in virtue of the relations (17) and (18), $m+2$ vanishing terms, it can have, by Descartes' Rule of Signs, at most $k-1$ positive roots. The assumptions $\Delta_1 \neq 0$ and $\Delta = 0$ are therefore inadmissible, i.e. if $\Delta_1 \neq 0$ then necessarily $\Delta \neq 0$.

Thus it has been shown that if no determinant of order $m+1$ formed from the matrix $M_m$ vanishes then no determinant of order $m+2$ formed from the matrix $M_{m+1}$ also vanishes. The proposition is therefore proved.

Now it is easy to show the lemma by aid of this proposition.

Since no determinant of order $m_k-k+1$ formed from the matrix

$$M_{m_k-k}=\begin{pmatrix} a_0, & a_1, \cdots\cdots, & a_{m_k} \\ a_{-1}, & a_0, \cdots\cdots, & a_{m_k-1} \\ \cdots & \cdots & \cdots \\ a_{-(m_k-k)}, & \cdots, & a \end{pmatrix},$$

vanishes we may eliminate from the system of equations

$$f(x)=a_0+a_1 x+\cdots\cdots+a_k x^k=0,$$
$$x f(x)=a_0 x+a_1 x^2+\cdots\cdots+a_k x^{k+1}=0,$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$x^{m_k-k} f(x)=a_0 x^{m_k-k}+\cdots\cdots+a_k x^{m_k}=0,$$

any $m_k-k$ terms and therefore those terms complementary to $x^{m_0}$, $x^{m_1}$, $\cdots\cdots$, $x^{m_k}$.

The result is

$$\psi(x)=A_0 x^{m_0}+A_1 x^{m_1}+A_2 x^{m_2}+\cdots\cdots+A_k x^{m_k}=0,$$

where $A_0$, $A_1$, $\cdots\cdots$, $A_k$ are constants expressible in the forms of determinants of order $m_k-k+1$ consisting of $m_k-k+1$ columns of the matrix $M_{m_k-k}$ and hence nonvanishing. This equation will have at least $k$ positive roots which are the roots of $f(x)=0$ and at most $k$ positive roots by Descartes' Rule of Signs, as it has $k+1$ nonvanishing terms. The positive roots of the two equations $\psi(x)=0$ and $f(x)=0$ will therefore coincide([1]).

By means of this lemma we shall now show the converse theorem above cited, namely that *if there exist a solution of the system of simultaneous equations* (11) *satisfying the conditions* (12) *then the system of trials belongs to the first class.*

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([1]) It follows that none of $A_0, A_1, \cdots, A_k$ vanishes or more strictly $A_0, A_1, \cdots, A_k$ have alternately positive and negative signs. It can also be shown that the equation $\psi(x)=0$ of such a property is uniquely determined, for if there exist two equations of the same property then the elimination of the term $x^{m_k}$ between them will give an equation having at most $k$ nonvanishing terms and at least $k$ positive roots which is impossible unless it becomes an identity.

First consider the case when none of $x_1, x_2, \cdots\cdots, x_r$ vanishes where $x_1, x_2, \cdots\cdots, x_r$ are, together with some properly chosen $\lambda_1, \lambda_2, \cdots\cdots, \lambda_r$, such as to satisfy the equations (12).

Let

$$f(x)=(x-x_1)^2(x-x_2)^2\cdots\cdots(x-x_r)^2,$$

and let

$$\psi(x)=A_0+A_1 x^{m_1}+\cdots\cdots+A_{2r-1} x^{m_{2r-1}}+x^{m_{2r}}$$

be a polynomial having the same positive zeros as $f(x)$ so that the function $\psi(x)$ has a definite sign positive for positive values of $x$ except for $x=x_1, x_2, \cdots\cdots, x_r$. Then considering the expression

$$\sum_{s=0}^{n} \psi\left(\frac{s}{n}\right) F(n, s),$$

instead of $\sum_{s=0}^{n} f\left(\frac{s}{n}\right) F(n, s)$ in § 3 in the Part I, and following the reasoning there adopted we may show that all the points in the interval $(0, 1)$ excepting $x_1, x_2, \cdots\cdots, x_r$ belong to the zero class.

Again let

$$f_i(x)=(x-x_1)(x-x_2)\cdots\cdots(x-x_r)/(x-x_i), \qquad i=1, 2, \cdots\cdots, r,$$

and let

$$\psi_i'(x)=A_0'+A_1' x^{m_1}+\cdots\cdots+A'_{r-2} x^{m_{r-2}}+x^{m_{r-1}}$$

be a polynomial having the same positive zeros as $f_i(x)$. Then taking

$$\sum_{s=0}^{n} \psi_i\left(\frac{s}{n}\right) F(n, s),$$

instead of $\sum_{s=0}^{n} f_i\left(\frac{s}{n}\right) F(n, s)$ in § 3 in the Part I, we may similarly prove that the point $x_i$ belongs to the first class, having the limiting probability $\lambda_i$.

Next consider the case when one of $x_1, x_2, \cdots\cdots, x_r$, say $x_1$, vanishes. In this case let

$$\psi(x)=A_2 x^{m_2}+A_3 x^{m_3}+\cdots\cdots+A_{2r-1} x^{m_{2r-1}}+x^{m_{2r}}$$

and

$$\psi_i(x) = A_1' x^{m_1} + A_2' x^{m_2} + \cdots\cdots + A'_{r-2} x^{m_{r-2}} + x^{m_{r-1}}$$

be two polynomials having the same positive zeros as

$$(x-x_2)^2 (x-x_3)^2 \cdots\cdots (x-x_r)^2,$$

and

$$(x-x_2)(x-x_3)\cdots\cdots(x-x_r)/(x-x_i), \qquad (i=2, 3, \cdots\cdots, r),$$

respectively. And considering the expressions

$$\sum_{s=0}^{n} \psi \binom{s}{n} F(n, s),$$

and

$$\sum_{s=0}^{n} \psi_i \binom{s}{n} F(n, s),$$

we see similarly that (i) all the points in the interval (0, 1) excepting 0, $x_2$, $x_3$, $\cdots\cdots$, $x_r$ belong to the zero class and (ii) the points $x_2$, $x_3$, $\cdots\cdots$, $x_r$ belong to the first class having the limiting probabilities $\lambda_2$, $\lambda_3$, $\cdots\cdots$, $\lambda_r$ respectively.

Then it readily follows, by aid of the relation

$$1 = \lambda_1 + \lambda_2 + \cdots\cdots + \lambda_r,$$

that the point $x=0$ also belongs to the first class having the limiting probability $\lambda_1$.

**3.** The result obtained in the last article may be stated as follows:—

*In order that a system of trials may be a simple determinate system of the first class with respect to an event, having r points of the first class it is necessary and sufficient that there should exist a solution of the simultaneous equations* (11) *with the conditions* (12).

It can however also be shown, as in § 4 in the Part I, that *the condition* (12) *may be replaced by*

$$x_i \neq x_k, \qquad (i \neq k), \tag{20}$$

$$\lambda_i \neq 0.$$

And hence we may also easily arrive at the following criterion:

*In order that a system of trials may be a simple determinate system of the first class having r points of the first class it is necessary and suf-*

*ficient that there should exist a unique solution of the simultaneous equations*

$$T_m = \sum_{i=1}^{r} x_i^m \lambda_i,$$

$$m = 0,\ m_1,\ m_2,\ \cdots\cdots,\ m_{2r}.$$

These results can readily be extended for a simple determinate system of non-independent trials of the first class with respect to a system of mutually exclusive events.

Tôkyô, May 1918.

# Sur l'équation $x_1^5+x_2^5+x_3^5+\cdots\cdots+x_k^5=0$,

par

A. Filippov, Taganrog, Russie.

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## Introduction.

Nous ne ferons pas une grande faute, sinons infirmerons que la théorie des nombres est enrichie des leur plusieures nouvelles théories grâce aux efforts, qui étaient faits par quelques mathématiciens, qui ont discuté le problème de Fermat. Quoique ce problème n'est pas encore résolu, sa discussion ne reste pas inutile pour un esprit fin.

Parmi beaucoup de travaux nouvaux il faut marquer ceux de M. Agronomof, qui sont d'autant plus remarquables, que les resultats bien substantiels sont obtenus par dés procédés les plus élémentaires. M. Agronomof discute l'équation indeterminée:

$$x_1^n+x_2^n+\cdots\cdots+x_h^n=y_1^n+y_2^n+\cdots\cdots+y_g^n.$$

En montrant les plus simples méthodes de la résolution de cette équation pour les valeurs de $h$ et $g$, surmontantes quelques limites, M. Agronomof pose le problème de leur abaissement.

Mais la nature des solutions pendant la diminution de nombre des inconnus se complique.

La question se presente par elle même, qu'on peut regarder comme une généralisation du problème de Fermat:

„Jusqu'aux quelques limites peut on diminuer les nombres des inconnus dans les équations de ce type pour elles restent resolubles en nombres entiers."

## 1. Deux catégories des solutions entières.

Considerons l'équation indeterminée

$$x_1^n + x_2^n + \cdots\cdots + x_h^n = y_1^n + y_2^n + \cdots\cdots + y_g^n. \tag{1}$$

On peut distribuer les solutions des équations de ce type en deux catégories. A la première catégorie se rapportent *les solutions sur zero:*

$$x_1 = x_2 = \cdots\cdots x_h = y_1 = y_3 = \cdots\cdots = y_g = 0. \tag{2}$$

A cette catégorie se rapportent aussi *les solutions symétriques.* Si $h = g$, nous avons:

$$x_1 = y_1 = a_1;\ x_2 = y_2 = a_2;\ \cdots\cdots;\ x_h = y_h = a_h. \tag{3}$$

Si $h \neq g$, nous avons *un système mêlé* des solutions. Par exemple l'équation $x_1^p + x_2^p + x_3^p = y_1^p + y_2^p$ a des solutions:

$$x_1 = 0, \quad x_2 = y_1 = a_1, \quad x_3 = y_2 = a_2. \tag{4}$$

Mais les autres équations de ce type ont les solutions de la *seconde catégorie.* Par exemple, l'équation

$$x^2 + y^2 = z^2 \tag{5}$$

a les solutions $x = 3$, $y = 4$, $z = 5$ ou $x = 5$, $y = 12$, $z = 13$.

Ainsi il est clair que, quand on parle, qu'une équation de ce type a les solutions, on suppose celles de seconde catégorie. Chaque solution générale peut être discuté à ce point de vue.

## 2. Équations du seconde, troisième et quatrième degré.

Nous savons que dans le cas du second degré la forme fermatique est résoluble en nombres entiers:

$$x^2 + y^2 = z^2. \tag{6}$$

L'équation du 3$^{me}$ degré peut être admise jusqu'au quatre membres:

$$x_1^3 + x_2^3 + x_3^3 + x_4^3 = 0. \tag{7}$$

Pour trouver les solutions de cette équation considerons d'abord l'équation:

$$x_1^3 + x_2^3 + x_2^3 + x_4^3 + x_5^3 = 0. \tag{8}$$

En suivant M. Agronomof, considerons l'identité:

$$(a_1+a_2+a_3)^3+(a_1-a_2-a_3)^3+(-a_1+a_2-a_3)^3$$
$$+(-a_1-a_2+a_3)^3=24\,a_1\,a_2\,a_3. \qquad (9)$$

En posant ici $a_1=9a^3$, $a_2=b^3$, $a_3=c^3$ on obtient pour la partie droite de cette identité l'expression :

$$24\,a_1\,a_2\,a_3=(6\,abc)^3. \qquad (10)$$

On en conclut que l'équation a les solutions :

$$\left.\begin{array}{ll} x_1=9a^3+b^3+c^3 & x_3=-9a^3+b^3-c^3 \\ x_2=9a^3-b^3-c^3 & x_4=-9a^3-b^3+c^3 \\ \multicolumn{2}{c}{x_5=-6\,abc.} \end{array}\right\} \qquad (11)$$

Si l'on prend $a=b=d$, $c=2d$, nous avons l'identité suivante :

$$(18d^3)^3+(-16d^3)^3+(-2d^3)^3+(-12d^3)^3=0. \qquad (12)$$

De là on tire, que l'équation a les solutions

$$x_1=9,\quad x_2=-8,\quad x_3=-1,\quad x_4=-6. \qquad (13)$$

Dans le cas du quatrième degré nous ne pouvons pas s'approcher si près de la forme fermatique. La plus simple équation du quatrième degré, autant que nous le savons, resoluble en nombres entiers est suivante :

$$x_1^4+x_2^4+x_3^4=y_1^4+y_2^4+y_3^4. \qquad (14)$$

Pour la résolution de cette équation prenons l'identité :

$$a_1^4+a_2^4+a_3^4-(a_1+a_2)^4-(a_1+a_3)^4-(a_2+a_3)^4+(a_1+a_2+a_3)^4$$
$$=12\,(a_1+a_2+a_3)\,a_1\,a_2\,a_3. \qquad (15)$$

Les paramètres $a_1$, $a_2$, $a_3$ il faut determiner de manière, que l'expression $12\,(a_1+a_2+a_3)\,a_1\,a_2\,a_3$ soit le quatrième degré d'un nombre entier.

Parce que $12\times108=6^4$, nous posons :

$$(a_1+a_2+a_3)\,a_1\,a_2\,a_3=108. \qquad (16)$$

D'un coup d'oeil nous apercevons la solution :

$$(1+2+6)\,1.2.6=108. \qquad (17)$$

De là, on tire

$$a_1=a, \quad a_2=2a, \quad a_3=6a. \tag{18}$$

Par conséquent nous avons trouvé l'identité:

$$a^4+(2a)^4+(9a)^4=(3a)^4+(7a)^4+(8a)^4. \tag{19}$$

Par suite l'équation (14) a les solutions:

$$\left.\begin{array}{lll} x_1=1, & x_1=2, & x_3=9 \\ y_1=3, & y_2=7, & y_3=8 \end{array}\right\} \tag{20}$$

## 3. Équation $x_1^5+x_2^5+\cdots\cdots+x_{17}^5=0$.

Dans son travail „Sur quelques équations du type $x_1^{2m+1}+x_2^{2m+1}+\cdots\cdots+x_k^{2m+1}=0$ résolubles en nombres entiers" (Izvestya Fisic. Matem. Obs. Kazan 1914) M. Agronomof a demontré la théorème suivante:

„L'équation $x_1^{2m+1}+x_2^{2m+1}+\cdots\cdots+x_k^{2m+1}=0$ est toujours résoluble en nombres entiers, si $k=4^n+1$ et $n\geqq m$." (I)

En posant ici $m=2$, on obtient une valeur minimum $k=17$.

En renvoyant le lecteur, qui s'interesse des formules générales et leur demonstrations, du travail indiqué, il faut remarquer que pour l'équation

$$x_1^5+x_2^5+\cdots\cdots+x_{17}^5=0. \tag{21}$$

Nous considérons l'identité:

$$\begin{aligned}&(a_1+a_2+a_3+a_4+a_5)^5-(-a_1+a_2+a_3+a_4+a_5)^5-(a_1-a_2+a_3+a_4+a_5)^5\\&-(a_1+a_2-a_3+a_4+a_5)^5-(a_1+a_2+a_3-a_4+a_5)^5-(a_1+a_2+a_3+a_4-a_5)^5\\&+(-a_1-a_2+a_3+a_4+a_5)^5+(-a_1+a_2-a_3+a_4+a_5)^5+(-a_1+a_2\\&+a_3-a_4+a_5)^5+(-a_1+a_2+a_3+a_4-a_5)^5+(a_1-a_2-a_3+a_4+a_5)^5\\&+\cdots\cdots=2^4 5!\, a_1 a_2 a_3 a_4 a_5.\end{aligned} \tag{22}$$

Nous discutons la partie droite de cette identité.

En posant ici

$$a_1=3^4 a^5, \quad a_2=2^3 b^5, \quad a_3=5^4 c^5, \quad a_4=d^5, \quad a_5=-e^5 \tag{23}$$

nous obtiendrons pour la partie droite l'expression:

$$2^4 5!\, a_1 a_2 a_3 a_4 a_5=-(60\, abcde)^5 \tag{24}$$

Il s'en suit, que l'équation (21) a les solutions:

$$\left.\begin{aligned}
x_1 &= 81a^5+8b^5+625c^5+d^5-e^5\\
x_2 &= 81a^5-8b^5-625c^5-d^5+e^5\\
x_3 &= -81a^5+8b^5-625c^5-d^5+e^5\\
x_4 &= -81a^5-8b^5+625c^5-d^5+e^5\\
x_5 &= -81a^5-8b^5-625c^5+d^5+e^5\\
&\cdots\cdots\cdots\cdots\cdots\cdots\\
x_{17} &= 60\,abcde.
\end{aligned}\right\} \qquad (25)$$

## 4. Équation $x^5+x_2^5+\cdots\cdots+x_{11}^5=0$.

La méthode de M. Agronomof nous permet d'abaisser la valeur de $k$ jusqu'au 11. Pour la résolution de l'équation

$$x_1^5+x_2^5+\cdots\cdots+x_{11}^5=0 \qquad (26)$$

M. Agronomof emploie la formule, dont il a donné la démonstration à son travail „Sur quelques classes des équations indeterminées résolubles en nombres entiers" (Iz. F. M. Ob. Kazan 1915).

Cette identité a l'expression:

$$T^1_{2m+2,\,2m+1}-T^2_{2m+2,\,2m+1}+\cdots\cdots+(-1)^{2m+1}\,T^{2m+2}_{2m+2,\,2m+1}=0. \qquad (27)$$

Ici $T^1_{2m+2,\,2m+1}$ est la somme des $2m+1$ puissances des $2m+2$ paramètres, $T^2_{2m+2,\,2m+1}$ est la somme des $2m+1$ puissances des sommes, qu'on peut composer des paramètres $a_1, a_2, a_3, \cdots\cdots, a_{2m+2}$, en les prenant par deux, etc.

Dans le cas $m=2$, nous avons:

$$\begin{aligned}
&a_1^5+a_2^5+a_3^5+a_4^5+a_5^5+a_6^5-(a_1+a_2)^5-(a_1+a_3)^5-(a_1+a_4)^5-(a_1+a_5)^5\\
&-(a_1+a_6)^5-(a_2+a_3)^5-(a_2+a_4)^5-(a_2+a_5)^5-(a_2+a_6)^5-(a_3+a_4)^5\\
&-(a_3+a_5)^5-(a_3+a_6)^5-(a_4+a_5)^5-(a_4+a_6)^5-(a_5+a_6)^5+(a_1+a_2+a_3)^5\\
&+(a_1+a_2+a_4)^5+\cdots\cdots\cdots\cdots-(a_1+a_2+a_3+a_4+a_5+a_6)^5=0. \qquad (28)
\end{aligned}$$

Dans la partie gauche de cette identité il y a 63 membres et par conséquent elle résolut l'équation:

$$x_1^5+x_2^5+x_3^5+\cdots\cdots+x_{63}^5=0.$$

En posant dans l'identité (28) $a_1=1$, $a_2=2$, $a_3=3$, $a_4=4$, $a_5=5$, $a_6=6$, on obtient l'identité:

$$1^5+2^5+8^5+9^5+10^5+15^5+(-5)^5+(-6)^5+(-7)^5+(-13)^5+(-14)^5=0$$

qui résolut l'équation (26).

## 5. Un système remarquable du troisième degré.

Dans son travail „ Sur quelques questions d'analyse indeterminée (The Tôhoku Math. Jour. 1916) M. Agronomof essaye encore d'abaisser les nombres des inconnus.

Il essaye démontrer le théorème suivant:

„ L'équation $x_1^n+x_2^n+\cdots\cdots x_h^n=y_1^n+y_2^n+\cdots\cdots+y_g^n$ peut être resolue en nombre entiers, si $h\geqq 2^{n-3}$, $g\geqq 2^{n-3}$, $n>4$ " (II)

Mais parce que M. Agronomof fait une faute caractéristique nous discutons cette question ici en détail.

M. Agronomof considère le système suivant:

$$\left.\begin{aligned} x_1^3+x_2^3+x_3^3+x_4^3&=y_1^3+y_2^3+y_3^3+y_4^3\\ x_1^2+x_2^2+x_3^2+x_4^2&=y_1^2+y_2^2+y_3^2+y_4^2\\ x_1+x_2+x_3+x_4&=y_1+y_2+y_3+y_4 \end{aligned}\right\} \tag{29}$$

qui a les solutions

$$\left.\begin{aligned} x_1&=b_1(a_1+a_2)+b_2(a_2+a_2)+b_3(a_3+a_4)+b_4(a_4+a_1)\\ x_2&=b_1(a_2+a_3)+b_2(a_3+a_4)+b_3(a_4+a_1)+b_4(a_1+a_2)\\ x_3&=b_1(a_3+a_4)+b_2(a_4+a_1)+b_3(a_1+a_2)+b_4(a_2+a_3)\\ x_4&=b_1(a_4+a_1)+b_2(a_1+a_2)+b_3(a_2+a_3)+b_4(a_3+a_4)\\ y_1&=b_1(a_1+a_2)+b_2(a_1+a_4)+b_3(a_3+a_4)+b_4(a_2+a_3)\\ y_2&=b_1(a_2+a_3)+b_2(a_1+a_2)+b_3(a_1+a_4)+b_4(a_3+a_4)\\ y_3&=b_1(a_3+a_4)+b_2(a_2+a_3)+b_3(a_1+a_2)+b_4(a_1+a_4)\\ y_4&=b_1(a_1+a_4)+b_2(a_3+a_4)+b_3(a_2+a_3)+b_4(a_1+a_2) \end{aligned}\right\} \tag{30}$$

où $b_1, b_2, b_3, b_4, a_1, a_2, a_3, a_4$, sont les nombres arbitrairement choisis.

Le système (29) jouit de cette propriété remarquable qu'on peut combiner par addition, soustraction, multiplication et division, tous les termes au premier degré, avec une indéterminée.

Par exemple, soit

$$x_1=45,\quad x_2=1,\quad x_3=24,\quad x_4=68,$$

$$y_1=66,\quad y_2=19,\quad y_3=3,\quad y_4=50.$$

Nous avons

$$45+1+24+68=66+19+3+50,$$

$$45^2+1^2+24^2+68^2=66^2+19^2+3^2+50^2,$$

$$45^3+1^3+24^3+68^3=66^3+19^3+3^3+50^3.$$

Alors nous avons aussi l'identité :

$$(90+z)+(2+z)+(48+z)+(136+z)$$
$$=(132+z)+(38+z)+(6+z)+(100+z),$$
$$(90+z)^2+(2+z)^2+(48+z)^2+(136+z)^2$$
$$=(132+z)^2+(38+z)^2+(6+z)^2+(100+z)^2$$
$$(90+z)^3+(2+z)^3+(48+z)^3+(136+z)^3$$
$$=(132+z)^3+(38+z)^3+(6+z)^3+(100+z)^3,$$

etc.

En multipliant les deux termes de l'identité dernière par $dz$ et integrant, nous obtiendrons une nouvelle identité du quatrième degré. En opérant de la même manière M. Agronomof a demontré le théorème suivant :

„ Le système

$$\left.\begin{array}{l} x_1^n+x_2^n+\cdots\cdots+x_{2n-1}^n=y_1^n+\cdots\cdots+y_{2n-1}^n \\ x_1^{n-1}+x_2^{n-1}+\cdots\cdots+x_{2n-1}^n=y_1^{n-1}+\cdots\cdots+y_{2n-1}^{n-1} \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ x_1+x_2+\cdots\cdots+x_{2n-1}=y_1+\cdots\cdots y_{2n-1} \end{array}\right\} \qquad \text{(III)}$$

on peut toujours résoudre en nombres entiers, si $a \geqq 3$."

Pour $n=5$ ce théorème nous donne la resolution de l'équation

$$x_1^5+x_2^5+\cdots\cdots+x_{32}^5=0.$$

Ainsi le théorème (I) donne la plus petite limite pour nombre des inconnus.

## 6. Équation $x_1^5+x_2^5+x_3^5+x_4^5=y_1^5+y_2^5+y_3^5+y_4^5$.

Considérons la methode de M. Agronomof et son théorème (II).

Prenons l'équation

$$x_1^5+x_2^5+x_3^5+x_4^5=y_1^5+y_2^5+y_3^5+y_4^5. \qquad (31)$$

Ici $n=5$, $h=4$, $g=4$.

Les solutions de l'équation (31) M. Agronomof definit par des substitutions :

$$\left.\begin{array}{llll} x_1=1+\kappa i_1, & x_2=1+\kappa i_2, & x_3=1+\kappa i_3, & x_4=1+\kappa i_4 \\ y_1=1+\kappa l_1, & y_2=1+\kappa l_2, & y_3=1+\kappa l_3, & y_4=1+\kappa l_4 \end{array}\right\} \quad (32)$$

Par suite

$$\kappa^5(i_1^5+i_2^5+i_3^5+i_4^5-l_1^5-l_2^5-l_3^5-l_4^5)+5\kappa^4(i_1^4+i_2^4+i_3^4+i_4^4-l_1^4-l_2^4-l_3^4-l_4^4)=0. \quad (33)$$

Pour la définition de $\kappa$ nous avons

$$\kappa=-\frac{5(i_1^4+i_2^4+i_3^4+i_4^4-l_1^4-l_2^4-l_3^4-l_4^4)}{(i_1^5+i_2^5+i_3^5+i_4^5-l_1^5-l_2^5-l_3^5-l_4^5)}. \quad (34)$$

Le cas $\kappa=0$ correspond au système symétrique

$$x_1=x_2=x_3=x_4=y_1=y_2=y_3=y_4=1.$$

Puis nous proposons que $\Sigma i^5-\Sigma l^5\neq 0$. Si $\Sigma i^5-\Sigma l^5=0$, le système (32) est en même temps le système des solutions de (29).

Considérons un exemple numérique

$$i_1=3,\ i_2=4,\ i_3=11,\ i_4=10,\ l_1=7,\ l_2=12,\ l_3=7,\ l_4=2. \quad (35)$$

Dans ce cas nous calculons

$$\kappa=-\frac{1}{7}.$$

D'où nous avons :

$$x_1=4,\quad x_2=3,\quad x_3=-4,\quad x_4=-3,$$

$$y_1=0,\quad y_2=-5,\quad y_3=0,\quad y_4=5.$$

Le système semblable est symétrique et appartient a la première catégorie. On peut démontrer, que la méthode de M. Agronomof *donne toujours pour l'équation* (31) *les solutions symétriques.*

## 7. Suite.

*D'abord* nous avons

$$i_1+i_3=i_2+i_4=l_1+l_3=l_2+l_4=(a_1+a_2+a_3+a_4)(b_1+b_2+b_3+b_4)=S. \quad (36)$$

La démonstration est évidente. Il faut additioner les formules correspondentes du système (30).

*En second,* nous avons :

$$i_1 i_3 + i_2 i_4 = l_1 l_3 + l_2 l_4. \tag{37}$$

Nous obtiendrons cette relation, si nous élevons à la seconde puissance les sommes $i_1+i_3$, $i_2+i_4$, $l_1+l_3$, $l_2+l_4$ et faisons toutes les transformations nécéssaires.

Maintenant nous pouvons calculer l'expression

$$S_4 = \Sigma i^4 - \Sigma l^4.$$

En élevant les mêmes sommes à la quatrième puissance, nous obtiendrons :

$$S_4 + 4i_1 i_3 (i_1^2 + i_3^2) + 4i_2 i_4 (i_2^2 + i_4^2) - 4l_1 l_3 (l_1^2 + l_3^2) - 4l_2 l_4 (l_2^2 + l_4^2)$$
$$+ 6i_1^2 i_3^2 + 6i_2^2 i_4^2 - 6l_1^2 l_3^2 - 6l_2^2 l_4^2 = 0$$

ou

$$S_4 + 4S^2(i_1 i_3 + i_2 i_4 - l_1 l_3 - l_2 l_4) - 2(i_1^2 i_3^2 + i_2^2 i_4^2 - l_1^2 l_3^2 - l_2^2 l_4^2) = 0. \tag{38}$$

Il s'en suit

$$S_4 = 2(i_1^2 i_3^2 + i_2^2 i_4^2 - l_1^2 l_3^2 - l_2^2 l_4^2). \tag{39}$$

De même

$$S_5 = 5(i_1 + i_3)(i_1^2 i_3^2 + i_2^2 i_4^2 - l_1^2 l_3^2 - l_2^2 l_4^2). \tag{40}$$

Ainsi

$$\kappa = -\frac{5S_4}{S_5} = \frac{-2}{i_1 + i_3}. \tag{41}$$

Maintenant il est clair que le système (32) des solutions de l'équation (31) est symétrique.

En effet

$$x_1 = 1 + \kappa i_1 = \frac{i_3 - i_1}{i_1 + i_3},$$

$$x_3 = 1 + \kappa i_3 = \frac{i_1 - i_3}{i_1 + i_3}.$$

De la suit $x_1 = -x_3$, $x_2 = -x_4$, $y_1 = -y_3$, $y_2 = -y_4$. Ainsi le théorème (II) de M. Agronomof n'est pas juste.

Il faut remarquer ici que d'autres formules de M. Agronomof § 4, § 5, § 6 „Sur quelqu. quest. etc." sont aussi illusoires et donnent les solutions de la première catégorie.

## 8. Les solutions entieres complexes.

Ainsi nous voyons que pour l'équation du cinquième degré nous pouvons abaisser le nombre des inconnus jusqu'au 11. C'est loin de la forme fermatique.

Le problème devient beaucoup plus facile s'il faut résoudre l'équation en nombres entiers complexes. Par exemple resolvons l'équation :

$$x_1^5+x_2^5+x_3^5+x_4^5+x_5^5=0. \tag{42}$$

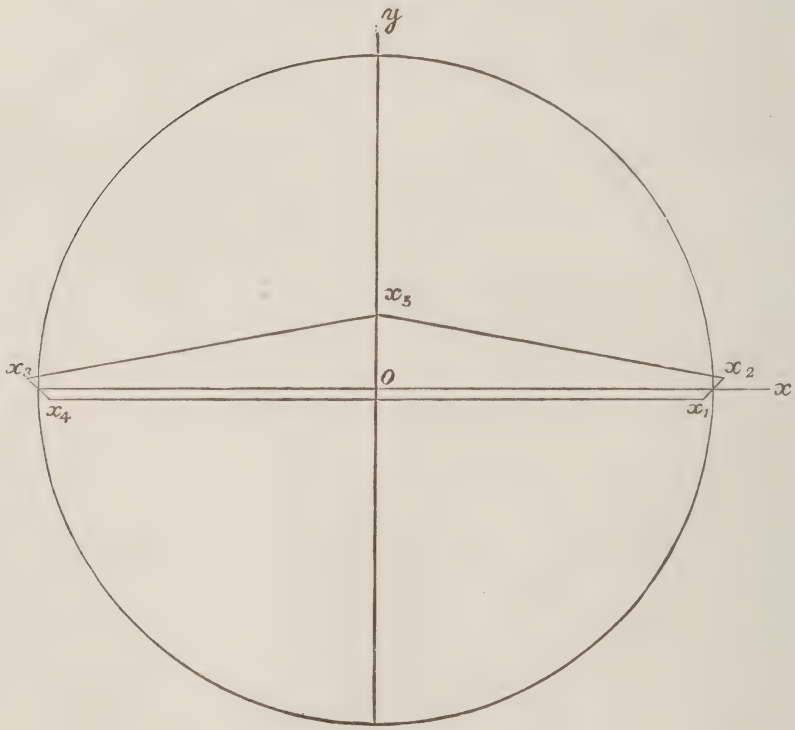


Fig. 1.

Pour notre but considerons l'identité :

$$(a_1+a_2+a_3)^5-(-a_1+a_2+a_3)^5-(a_1-a_2+a_3)^5-(a_1+a_2-a_3)^5$$
$$=80(a_1^2+a_2^2+a_3^2)\,a_1\,a_2\,a_3. \tag{43}$$

Posons

$$a_1 = 2^2 . 5^3 = 500, \quad a_2 = i^5, \quad a_3 = -1, \tag{44}$$

d'où nous avons :

$$80(a_1^2 + a_2^2 + a_3^2)\, a_1 a_2 a_3 = -2^{10} 5^{10} i^5. \tag{45}$$

Ainsi nous voyons que l'équation a les solutions :

$$x_1 = a_1 + a_2 + a_3 = 499 - i,$$

$$x_2 = a_1 - a_2 - a_3 = 501 + i,$$

$$x_3 = -a_1 + a_2 - a_3 = -499 - i,$$

$$x_4 = -a_1 - a_2 + a_3 = -501 + i,$$

$$x_5 = 100i;$$

ici nous prenons une seule valeur $\sqrt[5]{1} = 1$.

La construction de ce système donne une pentagone symétrique (*Fig. 1*).

Reval-Taganrog, Mai-Decembre 1917.

# On Closed Curves Described by a Spherical Pendulum,

by

Arnold Emch, Urbana, Ill., U. S. A.

## I. Introduction.

Treatises on elliptic functions and their applications([1]) contain without exception a more or less extended discussion of the motion of a spherical pendulum. It is usually shown how $z$, the distance of the pendulum-bob from the plane of suspension, may be represented as an elliptic function of the time $t$, and how the angle $\theta$ which the horizontal projection of the pendulum makes with the $x$-axis, and the coordinates $x$ and $y$ may be expressed by Weierstrassian $\sigma$-functions and elliptic functions of the second kind.

Thus, from the differential equations of the motion:

$$(1) \qquad \frac{d^2x}{dt^2} = -S\frac{x}{l}, \quad \frac{d^2y}{dt^2} = -S\frac{y}{l}, \quad \frac{d^2z}{dt^2} = -S\frac{z}{l} - g,$$

in which $l$ is the length of the pendulum, $S$ the reaction directed towards the origin (point of suspension), the velocity $v$ of the pendulum-bob is obtained as $v = \sqrt{h - 2gz}$. The constant $h$ depends upon the initial conditions. For $z$ as a function of $t$, there is

$$(2) \qquad z = \frac{h}{6g} + \frac{2l^2}{g}\wp(t + \omega_2).$$

In this Weierstrassian $\wp$-function the one halfperiod $\omega_1$ is real; the other, $\omega_2$ is purely imaginary. Putting $u = t + \omega_2$, let $a$ and $b$ be the arguments for which $z$ assumes the values $-l$ and $+l$, respectively,

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([1]) See for example:

Alfred George Greenhill: *The Applications of Elliptic Functions*, pp. 214–219.

Jules Tannery et Jules Molk: *Éléments de la Théorie des Fonctions Elliptiques*, Vol. IV, 176–192.

P. Appell et E. Lacour: *Principes de la Théorie des Fonctions Elliptiques*, pp. 90–102.

Heinrich Burkhardt: *Elliptische Funktionen*, pp. 353–363.

Robert Fricke: *Analytisch-Funktionentheoretische Vorlesungen*, pp. 315–322.

and $t_1$ and $t_2$ the corresponding complex values of $t$. It is found that $a=i\alpha$, where $\alpha$ is real and positive, and $b=\omega_1+i\beta$, with $\beta$ equal to a fraction of $|\omega_2|$. The constant of integration may be determined such that $x=r_0$, $y=0$, when $t=0$. Under these conditions $\theta$ is defined by

$$(3) \qquad e^{2i\theta}=\frac{\sigma(t+t_1)\,\sigma(t+t_2)}{\sigma(t-t_1)\,\sigma(t-t_2)}e^{-2t\{\zeta_2(t_1)+\zeta_2(t_2)\}}.$$

For $x$ and $y$ we get

(4)

$$x=\frac{r_0}{2}\cdot\frac{\sigma(t+t_1)\sigma(t+t_2)e^{-t\{\zeta_2(t_1)+\zeta_2(t_2)\}}+\sigma(t-t_1)\sigma(t-t_2)e^{t\{\zeta_2(t_1)+\zeta_2(t_2)\}}}{\sigma(t_1)\,\sigma(t_2)\,\sigma_2^2(t)},$$

(5)

$$y=-i\frac{r_0}{2}\cdot\frac{\sigma(t+t_1)\sigma(t+t_2)e^{-t\{\zeta_2(t_1)+\zeta_2(t_2)\}}-\sigma(t-t_1)\sigma(t-t_2)e^{t\{\zeta_2(t_1)+\zeta_2(t_2)\}}}{\sigma(t_1)\,\sigma(t_2)\,\sigma_2^2(t)}.$$

These solutions for $x$ and $y$ may also be obtained from the following differential equations:

$$(6) \qquad \frac{d^2x}{dt^2}=\left[6\,\wp(t)-\frac{h}{2l^2}\right]x,$$

and a similar equation for $y$, which are known as particular cases of Lamé's differential equation. For real value of $t, x$ and $y$ assume real values and are uniform functions of $t$ with $\infty$ as the only essential singularity.

The problem to establish serial expansions for $x$ and $y$ in case of a general periodic closed pendulum motion was solved by Professor F. R. Moulton, in a paper published in the Rendiconti del Circolo Matematico di Palermo (1).

So far as the writer is aware the geometrical aspect of the problem of closed curves in a pendulum motion has never been fully discussed, and it is the object of this paper to establish some of the geometric properties of these curves.

## II. General Case of Closed Curves.

Writing for abbreviation

(1) Vol. XXXII, pp. 338–364 (1911).

$$(7)\qquad \frac{r_0}{2}\,\frac{\sigma(t+t_1)\sigma(t+t_2)e^{-t\{\zeta_2(t_1)+\zeta_2(t_2)\}}}{\sigma(t_1)\,\sigma(t_2)\,\sigma_2^2(t)}=f(t),$$

we can write (4) and (5) in the form

$$(8)\qquad x=\phi(t)=f(t)+f(-t),$$

$$(9)\qquad y=\psi(t)=-i\{f(t)-f(-t)\}.$$

Increasing $t$ by $2m_1\,\omega_1$, a multiple of the real period, we get for $x$ and $y$ the new values

$$(10)\qquad x_{m_1}=f(t)\,e^{2m_1\{\eta_1(a+b)-\omega_1[\zeta(a)+\zeta(b)]\}}+f(-t)\,e^{-2m_1\{\eta_1(a+b)-\omega_1[\zeta(a)+\zeta(b)]\}},$$

$$(11)\qquad y_{m_1}=-i\{f(t)\,e^{2m_1\{\eta_1(a+b)-\omega_1[\zeta(a)+\zeta(b)]\}}+f(-t)\,e^{-2m_1\{\eta_1(a+b)-\omega_1[\zeta(a)+\zeta(b)]\}}\}.$$

Since $a=i\alpha$, $b=\omega_1+i\beta$ ($\alpha, \beta$ real and positive) we find by well known formulas of elliptic functions that

$$\eta_1(a+b)-\omega_1\{\zeta(a)+\zeta(b)\}$$
$$=2\omega_1\,\zeta(i\alpha)-2i\alpha\,\zeta(\omega_1)+\frac{\omega_1}{2}\,\frac{\wp'(i\alpha)}{\wp(\omega_1)-\wp(i\alpha)},$$

which is a purely imaginary quantity.

It is therefore possible to determine the constants $a, b, \omega_1$, or the constants of motion, in such a manner that

$$(12)\qquad 2m_1\{\eta_1(a+b)-\omega_1[\zeta(a)+\zeta(b)]\}=2ki\pi,$$

where $k$ is a positive integer.

In this case the motion is apparently periodic, since $x_{m_1}=x$, $y_{m_1}=y$, and, of course, $z_{m_1}=z$. If in the expression (3) defining $\theta$, $t$ is increased by $2\omega_1$, it is easily found that $\theta$ will increase by the amount

$$(13)\qquad \Phi=\frac{2}{i}\{\eta_1(a+b)-\omega_1[\zeta(a)+\zeta(b)]\},$$

so that, according to (12)

$$(14)\qquad \Phi=\frac{2k\pi}{m_1}.$$

From this is seen that the angle $\Phi$ must be described $m_1$ times before the pendulum reaches its original position. During this time it will have described $k$ complete revolutions about the vertical axis. From the

formula (2) for $z$ it is seen that $z$ has two zeros within the period-interval $2\omega_1$ for $z$; consequently $2m_1$ zeros within the period $2m_1\omega_1$ of the closed motion.

*The path described by the pendulum is a closed curve intersecting every level between lowest and highest positions in $2m_1$ points and winding $k$ times around the z-axis.*

We shall now investigate the character of the horizontal projection of this curve, whose parametric equations are

$$x=\phi(t), \quad y=\psi(t). \tag{15}$$

Consider any straight line

$$\alpha x+\beta y+\gamma=0. \tag{16}$$

The intersections of this line with the curve are evidently determined by the zeros of the function

$$F(t)=\alpha\cdot\phi(t)+\beta\cdot\psi(t)+\gamma. \tag{17}$$

This is a uniform and simply periodic function of $t$ with the period $2m\omega_1$, which assumes real values for real values of $t$, and which has $\infty$ as the only essential singularity. It can therefore have only a finite number of real zeros within the period-interval, i. e. the line (16) cuts the curve in a finite number of real points only. If we also admit complex values for $t$, within the period strip $P$ of width $2m_1\omega_1$ parallel to the axis of purely imaginary $t$'s, also the number of complex zeros within the finite region of $P$ must be finite, since otherwise $F(t)$ would have an essential singularity in this finite region. Hence the assumption of an infinite number of zeros $t_j$, $j=1, 2, 3.\cdots\cdots$, requires that these accumulate in the neighborhood of $t=\infty$, outside of the real axis, and in the infinite region of $P$. The set of complex values $(x_j, y_j)$ of the points of intersection of the line (16) with the curve, which correspond to the set of zeros $t_j$ cannot approach a finite value $(x_0, y_0)$, or point, since in this case there would exist a infinite set of finite corresponding zeros $t_j'$ which would necessarily accumulate around a finite point $t_0$ as a limit. This point $t_0$ would therefore be an essential singularity of $F(t)$, which is not possible. Hence for $t_j=\infty$ within $P$ both $x_j$ and $y_j$ must approach infinity and at the same time satisfy

$$\alpha x_j+\beta y_j+\gamma=0.$$

From this follows that

$$\lim_{t_j=\infty}\left(\frac{y_j}{x_j}\right)=-\frac{\alpha}{\beta},$$

a definite real constant.

In order to see whether this limit is possible, write $t$ in the form $t=\tau+2m_2\omega_2$, where $\tau$ is a finite point in $P$. Putting

$$\sigma(t+t_1)\,\sigma(t+t_2)\,e^{-t\{\zeta_2(t_1)+\zeta_2(t_2)\}}=f_1(t),$$

we get

$$M(t)=\frac{y}{x}=-i\frac{f_1(t)-f_1(-t)}{f_1(t)+f_1(-t)}.$$

For $t=\tau+2m_2\omega_2$ there is

(18) $$M(\tau+2m_2\omega_2)=-i$$
$$\times\frac{f_1(\tau)\exp\left[2m_2\{\eta_2(a+b)-\omega_2[\zeta(a)+\zeta(b)]\}\right]-f_1(-\tau)\exp\left[-2m_2\{\eta_2(a+b)-\omega_2[\zeta(a)+\zeta(b)]\}\right]}{f_1(\tau)\exp\left[2m_2\{\eta_2(a+b)-\omega_2[\zeta(a)+\zeta(b)]\}\right]+f_1(-\tau)\exp\left[-2m_2\{\eta_2(a+b)-\omega_2[\zeta(a)+\zeta(b)]\}\right]}.$$

For $\tau=\omega_2$, $x$ and $y$ both become infinite, and $f_1(\tau)$ and $f_1(-\tau)$ remain finite. There is

$$2m_2\{\eta_2(a+b)-\omega_2[\zeta(a)+\zeta(b)]\}$$
$$=2m_2\left\{2\omega_2\zeta(ia)-2\eta_2\cdot ia+\frac{\omega_2}{2}\,\frac{\wp'(ia)}{\eta_1-\wp(ia)}+\omega_2\eta_1-\eta_2\omega_1\right\}=m_2K_2+m_2\pi i,$$

where $K_2$ is real. Consider now the expressions

$$A_2=\exp[m_2K_2+m_2i\pi]=(-1)^{m_2}\cdot\exp[m_2K_2],$$

and

$$B_2=\exp[-m_2K_2-m_2i\tau]=(-1)^{m_2}\cdot\exp[-m_2K_2].$$

Suppose that $t_j$ approaches infinity in the neighborhood of a pole of $x$ and $y$, then $\tau\doteq\omega_2$ and $m_2\doteq\pm\infty$. In case that $K_2>0$ and $m_2\doteq+\infty$, $A_2\doteq\pm\infty$, $B_2\doteq0$, and consequently $\lim M(t)=-i$. In case that $K_2>0$, and $m_2\doteq-\infty$, $A_2\doteq0$, and $B_2\doteq\pm\infty$, and $\lim M(t)=+i$.

In case that $K_2<0$ and $m_2\doteq+\infty$, $A_2\doteq0$, $B_2\doteq\pm\infty$ and $\lim M(t)=+i$. When $K_2<0$ and $m_2\doteq-\infty$, $A_2\doteq\pm\infty$, $B_2\doteq0$, and $\lim M(t)=-i$.

Next assume the possibility that $\tau$ approaches one of the points $t_1, t_2, -t_1, -t_2$, and again $m_2\doteq\pm\infty$:

First let $\tau\doteq t_1$, when $K_2>0$ and $m_2\doteq+\infty$, then again $\lim M(t)$. $=-$

For $\tau\doteq t_1$, when $K_2>0$ and $m_2\doteq-\infty$, $\lim M(t)=+i$.

For $\tau \doteq t_1$, when $K_2 < 0$ and $m_2 \doteq +\infty$, $\lim M(t) = +i$.

For $\tau = t_1$, when $K_2 < 0$ and $m_2 \doteq -\infty$, $\lim M(t) = -i$.

In a similar manner it can be shown that also the remaining 12 cases of this kind and all other points congruent to $t_1, t_2, -t_1, -t_2$ within $P$ tend to $\lim M(t) = \pm i$, for $m_2 \doteq \pm\infty$.

Finally assume that $\tau$ be any point different from those considered before, then for $t = \tau + 2m_2\omega_2$, $f_1(\tau)$ and $f_1(-\tau)$ are finite and not vanishing, so that for $m_2 \doteq \pm\infty$, $\lim M(t) = \pm i$. This shows conclusively that under the stated conditions $\lim\left(\frac{y}{x}\right) = \lim M(t) = \pm i$ cannot be equal to $-\frac{\alpha}{\beta}$, and that consequently $F(t)$ has no zeros at infinity and the curve cannot intersect a real line in infinite points. The number of intersections with any real line is therefore finite. On the other hand, for an isotropic line $x \pm iy + \gamma = 0$, the above limits show, that the curve passes through the isotropic points. We may therefore state the

*Theorem: The horizontal projection of a closed pendulum-curve, and consequently the pendulum-curve itself are algebraic curves. The horizontal projection passes through the isotropic (circular) points.*

When the period of the closed motion is $2m_1\omega_1$, the curve consists of $m_1$ loops. Suppose that $\tau_1$ and $\tau_2$ $(\tau_2 > \tau_1)$ are two consecutive values of $t$, determining two points on a branch of such a loop, and $\theta_1$ and $\theta_2$ the corresponding polar-angles. Then from (3) we get

$$(19) \quad 2i(\theta_2 - \theta_1)$$

$$= \log \frac{\sigma(\tau_2+t_1)\,\sigma(\tau_2+t_2)\,\sigma(\tau_1-t_1)\,\sigma(\tau_1-t_2)}{\sigma(\tau_2-t_1)\,\sigma(\tau_2-t_2)\,\sigma(\tau_1+t_1)\,\sigma(\tau_1+t_2)} - 2\{\zeta_2(t_1)+\zeta_2(t_2)\}(\tau_2-\tau_1).$$

Since $\frac{\sigma(t+t_1)\,\sigma(t+t_2)}{\sigma(t-t_1)\,\sigma(t-t_2)}$ is an elliptic function with the real period $2\omega_1$, we find for the arguments $\tau_1' = 2m_1\omega_1 - \tau_1$, $\tau_2' = 2m_1\omega_1 - \tau_2$ the corresponding angles $\theta_1'$ and $\theta_2'$ and from (3) in comparison with (19)

$$\theta_2' - \theta_1' = \theta_1 - \theta_2 = -(\theta_2 - \theta_1).$$

The values of $z_1'$ and $z_2'$ for $\tau_1'$ and $\tau_2'$ are the same as those of $z_1$ and $z_2$ for $\tau_1$ and $\tau_2$, respectively. From this follows, as is well known, that each loop consists of two symmetric branches, as indicated in Fig. 1 for the general case, and in Fig. 2 for the special case to be discussed in the next section.

*The closed algebraic curve has therefore an $m_1$-fold axial symmetry.*

## III. Closed Curves in Greenhill's Case.

In Greenhill's case[1] the pendulum-bob reaches (but does not go above) the horizontal plane of suspension with a non-vanishing horizontal velocity, and the parametric expressions for the horizontal projection may be written in the form[2]

$$(20)\quad x=r_0\left[\cos(A+\pi)\,v\cdot\mathrm{cn}\,(2Kv)-\mu\sin(A+\pi)\,v\cdot\mathrm{sn}\,(2Kv)\,\mathrm{dn}\,(2Kv)\right],$$

$$(21)\quad y=r_0\left[\sin(A+\pi)\,v\cdot\mathrm{cn}\,(2Kv)+\mu\cos(A+\pi)\,v\cdot\mathrm{sn}\,(2Kv)\,\mathrm{dn}\,(2Kv)\right].$$

The coordinate $z$ becomes

$$(22)\qquad z=a\,\mathrm{cn}^2\,(2Kv)\,,$$

where $a$ is the greatest root in the cubic equation entering the problem of the spherical pendulum. The next root $b=0$ in this case. The quantities $a$ and $b$ are different from those denoted by the same letters in the foregoing sections. $4K$ is the real period in the Jacobian elliptic functions;

$$v=\frac{t}{2\omega_1}\,,\qquad A+2\pi=\Phi=\frac{2k\pi}{m_1}$$

in a closed motion;

$$\mu=\sqrt{(l^2+a^2)/(l^2-a^2)}\,.$$

There is

$$A+\pi=\frac{2k-m_1}{m_1}\,\pi\,.$$

The common period for the elliptic functions in (20), (21), (22) is $4K$, that of the cosine and sine is $m_1$ when $2k-m_1$ is even, $2m_1$ when $2k-m_1$ is odd. The common period of $x, y, z$ is $m_1$. When $m_1$ is even this is apparent; when $m_1$ is odd, it follows from the fact, that

$$\cos(u+(2k-m_1)\,\pi)=-\cos u,$$

$$\sin(u+(2k-m_1)\,\pi)=-\sin u\,;$$

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(1) Greenhill: *Les Fonctions Elliptiques et leurs Applications*, Chap. III.

Appell: *Traité de Mecanique Rationnelle*, t. 1, p. 494.

Emch: *A Theorem on the Curves Described by a Spherical Pendulum*, Bull. Amer. Math. Society, Vol. 23, pp 230–232 (1917).

(2) Tannery and Molk, loc. cit., p. 188. Here the positive direction of the $z$-axis is downward.

$$\operatorname{cn}(u+2m_1 K) = -\operatorname{cn} u,$$
$$\operatorname{sn}(u+2m_1 K) = -\operatorname{sn} u,$$
$$\operatorname{dn}(u+2m_1 K) = -\operatorname{dn} u.$$

It is well known that in a non-singular spherical pendulum-motion there is always

$$\pi < \Phi < 2\pi .$$

On arc of the curve corresponding to an increase of $\theta$ by $2\pi$ cuts a line through the origin, for instance the $x$-axis in two points only. The end-point of this arc must be counted as the initial point of the next turn through $2\pi$. As the curve consists of $m_1$ arcs of this sort, it will cut a real line at most in $2k$ points. There are clearly always lines for which this number is attained.

That the curve is algebraic follows from the fact established before, that every closed pendulum curve is algebraic. This can also be proved independently as follows.

In (20) and (21) put

$$v = m_1 v', \qquad \frac{2k-m_1}{m_1}\pi v' = \xi, \qquad 2Kv' = \eta,$$

then

$$x = r_0 [\cos(m_1 \xi) \cdot \operatorname{cn}(m_1 \eta) - \mu \sin(m_1 \xi) \cdot \operatorname{sn}(m_1 \eta) \cdot \operatorname{dn}(m_1 \eta)],$$
$$y = r_0 [\sin(m_1 \xi) \cdot \operatorname{cn}(m_1 \eta) + \mu \cos(m_1 \xi) \cdot \operatorname{sn}(m_1 \eta) \cdot \operatorname{dn}(m_1 \eta)].$$

Next, put

$$\sin\xi = x_1, \quad \cos\xi = x_2, \quad \operatorname{sn}\eta = x_3, \quad \operatorname{cn}\eta = x_4, \quad \operatorname{dn}\eta = x_5.$$

Now it is well known that $\sin(m_1\xi)$ and $\cos(m_1\xi)$ may be expressed as polynomials of $x_1$ and $x_2$; likewise, $\operatorname{sn}(m_1\eta)$, $\operatorname{cn}(m_1\eta)$, $\operatorname{dn}(m_1\eta)$ as rational functions of $x_3, x_4, x_5$. Hence, $x$ and $y$ may be expressed as rational functions of $x_1, x_2, x_3, x_4, x_5$. On the other hand

$$\sin^2\xi + \cos^2\xi = 1,$$
$$\operatorname{cn}^2\eta + \operatorname{sn}^2\eta = 1,$$
$$\operatorname{dn}^2\eta + k^2\operatorname{sn}^2\eta = 1.$$

Moreover, as

$$x^2 + y^2 + z^2 = l^2,$$
$$r_0^2 [\operatorname{cn}^2\eta + \mu^2 \operatorname{sn}^2\eta \operatorname{dn}^2\eta] + a^2 \operatorname{cn}^4\eta = l^2.$$

We have therefore six rational integral equations in the $x_i$'s ($i=1, 2, 3, 4, 5$)

$$R_x(x_1, x_2, x_3, x_4, x_5; x)=0,$$
$$R_y(x_1, x_2, x_3, x_4, x_5; y)=0,$$
$$x_1^2+x_2^2-1=0,$$
$$x_3^2+x_4^2-1=0,$$
$$k\,x_5^2+x_5^2-1=0,$$
$$r_0^2\,\mu^2\,x_3^2\,x_5^2+r_0^2\,a^2\,x_4^2+r_0^2+x_4^2-l^2=0,$$

in the first two of which $x$ and $y$ enter into the coefficients. The resultant([1]) of these is a rational function of the coefficients, so that $x$ and $y$ are connected by an algebraic equation.

## Nature of the Curve and Degree of its Equation.

Putting again $x=\phi(t)$, $y=\psi(t)$, and

$$(23)\qquad \begin{cases} x'=x\cos(A+\pi)\,v+y\sin(A+\pi)\,v, \\ y'=-x\sin(A+\pi)\,v+y\cos(A+\pi)\,v, \\ z'=z, \end{cases}$$

from (20) and (21) we find

$$(24)\qquad \begin{cases} x'=r_0\,\mathrm{cn}\,(2Kv), \\ y'=\mu\,r_0\,\mathrm{sn}\,(2Kv)\,\mathrm{dn}\,(2Kv), \\ z'=a\,\mathrm{cn}^2\,(2Kv). \end{cases}$$

From (24) follows that the locus of the point $P'(x', y', z')$ is a quartic $C_4^{*\prime}$ in space, and from (23), that the point $P'$ is obtained by rotating the point $P(x, y, z)$ of the pendulum curve $C$ about the $z$-axis through an angle $-(A+\pi)\,v$, i.e. through an angle negatively proportional to the time associated with $P$ in the motion. The curve $C_4^{*\prime}$ consists of a single real closed branch([2]), whose horizontal projection on the $XY$-plane has the equation

$$\mu^2\,k^2\,x'^4+\mu^2\,r_0^2\,(1-2\,k^2)\,x'^2+r_0^2\,y'^2-\mu^2\,r_0^4\,(1-k^2)=0.$$

Since in Greenhill's case

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([1]) Pascal's *Repertorium* (I, first half) 2nd ed, p. 273.

([2]) Loc. cit, Bulletin of the American Mathematical Society.

$$k^2 = a^2/(l^2+a^2), \qquad \mu^2 = (l^2+a^2)/(l^2-a^2),$$

this equation reduces to

$$(25) \qquad a^2 x'^4 + r_0^2 (l^2-a^2)(x^2+y^2) - r_0^4 l^2 = 0.$$

This is a plane quartic $C_4'$ of deficiency 1, with a real finite branch without inflexions, and the infinite point of the $y$-axis as a real isolated point, a tacnode where two imaginary branches touch each other. It has clearly no finite closed imaginary branch.

Consider next the horizontal projection $C$ of the pendulum curve as parametrically represented by (20) and (21). A straight line in the same plane cuts it in at most $2k$ real points, so that its order is not less than $2k$. Supposing the order to be greater than $2k$, then the imaginary intersections of the curve with the $y$-axis, for example, must be even in number. If $\alpha+i\beta$ is the ordinate of such a point, then $\alpha-i\beta$ is the ordinate of another imaginary point of the curve on the $y$-axis. But the curve is symmetric with respect to the $x$-axis, so that also $-(\alpha+i\beta)$ and $-(\alpha-i\beta)$ are the ordinates of such points. From the form of (20) and (21) and the nature of the functions making up these expressions follows that for $v=\pm i\tau$, $y=\pm ie$ ($e$ real and $>0$) and $x$ real and $>0$, so that points of the curve on the $y$-axis ($x=0$, $y=\pm ie$) are not possible. Hence when there are any imaginary intersection of $C$ and the $y$-axis at all there are at least few of them, so that the order of the curve would be at least $2k+4$. Suppose that $t_1$ is a parameter for which

$$x=\phi(t_1)=0, \qquad y=\psi(t_1)=\alpha+i\beta$$

represent an imaginary point of $C$ on the $y$-axis. Then the point with the coordinates

$$x=\phi(t_1+t), \qquad y=\psi(t_1+t)$$

describes a closed imaginary branch $C_1$ of $C$, when $t$ increases through real values from 0 to $2m_1\omega_1$, while at the same time the angle $\theta$ increases from $\theta_1=\frac{\pi}{2}$ to $\frac{\pi}{2}+2k\pi$. For the values $t_1+t$ of the parameter between the given limits, $x'$ and $y'$ in (23) remain finite, so that by (23) the branch $C_1$ would be transformed into a closed imaginary branch of the quartic $C$, which is not possible. The curve $C$ consists therefore of one real closed branch only, with no separate imaginary branch, and

is therefore rational([1]), and of order $2k$. Hence the

*Theorem: The projection of a closed pendulum curve upon the horizontal plane of suspension in Greenhill's case is a rational algebraic curve.*

*If the angle corresponding to the period $2\omega_1$ is $2k\pi/m_1$, the degree of the curve is $2k$.*

## IV. Classification of Curves of Type $C$.

Let $C$ be the projection upon the $XY$-plane of a closed pendulum curve of Greenhill's type of order $2k$ with an $m_1$-fold axial symmetry, so that the angle $\Phi$ corresponding to the period $2\omega_1$ is $\Phi = 2k\pi/m_1$. Now it is known that in any pendulum motion $2\pi > \Phi > \pi$, so that, $k$ and $m_1$ being relatively prime,

$$\frac{m_1}{2} < k < m_1. \tag{26}$$

Hence when $m_1$ is odd, $k$ may have the values

$$k = \frac{m_1+1}{2},\ \frac{m_1+3}{2},\ \frac{m_1+5}{2},\ \ldots\ldots,\ \frac{m_1+2(m_1-3)/2+1}{2}, \tag{27}$$

so that for a given odd $m_1$ there are $\frac{m_1-3}{2}+1 = \frac{m_1-1}{2}$ curves of the type $C$. $m_1\frac{\Phi}{2}$ and $m_1\Phi$ are values of $\theta$ for which the curve cuts the $x$-axis of symmetry orthogonally. As there are the only values of this kind, these points of intersection on the $x$-axis are not double-points, and are two in number. All other intersections of $C$ with the $x$-axis are double-points. There are therefore $k-1$ double points on every axis of symmetry.

Hence the

*Theorem: Every curve of type $C$ of order $2k$, and $m_1$-fold symmetry, has $m_1(k-1)$ real double-points.*

This is, of course, also true, when $m_1$ is even. In order to determine

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([1]) Harnack: *Ueber die Vieltheiligkeit der ebenen algebraischen Curven*, Math. Annalen, Vol. 10, pp. 189–198.

Hilbert; *Ueber die reellen Züge algebraischer Curven*, Math. Annalen, Vol. 38, pp. 115–138.

whether among the $\frac{m_1-1}{2}$ rational curves there are curves with all double-points real, let

$$k=(m_1+2m_2+1)/2, \qquad m_2 \leqq (m_1-3)/2,$$

when $m_1$ is odd, be the degree of any of these curves. It will have

$$m_1(k-1)=m_1\,\frac{m_1+2m_2-1}{2}$$

real double points.

On the other hand, when the order of the curve is $2k$, the total number of double-points is $(2k-1)(2k-2)/2$. Hence when all double points are real we have the condition

$$m_1(m_1+2m_2-1)/2=(m_1+2m_2)(m_1+2m_2-1)/2.$$

This reduces to $m_2(2m_2+m_1-1)=0$. Consequently the maximum number of real double-points is attained, when either $m_2=0$, or $m_2=(1-m_1)/2$. But if $m_1>1$, we would have $m_2<1$, which is not possible. The only possibility is therefore $m_2=0$, so that there is only one curve of the set for which all double-points are real. Its degree is clearly $2k=m_1+1$, and the number of double-points $(m_1-1)m_1/2$.

When $m_1$ is even we have for $k$ the possibilities:

(28) $k=(m_1+2)/2,\ (m_1+4)/2,\ (m_1+6)/2, \cdots\cdots,\ \{m_1+2(m_1-2)/2\}/2,$

i.e. altogether $(m_1-2)/2$ cases. For any $k$ of this set we have

$$k=(m_1+2m_2)/2, \qquad m_2 \leqq (m_1-2)/2.$$

The condition for all double-points real is now

$$m_1(m_1+2m_2)/2-1=(m_1+2m_2-1)(m_1+2m_2-2)/2,$$

which reduces to

$$4m_2^2+2m_2(m_1-3)-m_1+2=0.$$

From this equation we get for $m_2$ the two values

$$m_2=(2-m_1)/2, \text{ and } m_2=1/2.$$

As $m_1>2$, there are consequently no curves of the set when $m_1$ is even, for which all double-points are real.

The following tables contain a classification of curves $C$ according to the values of $k$ and $m_1$, first when $m_1$ is odd (type A), second when $m_1$ is even (type B). The columns contain the designation and the characteristic constants, the row the various curves. The table for $m_1$ odd has been carried as far as $m_1=11$, that for $m_1$ even as far as $m_1=12$. Both may be extended indefinitely without difficulty. It will be proved later that the imaginary double-points are all absorbed by the circular points at infinity.

A. $m_1$ odd.

| Designation | $m_1$ | Order $=2k$ | Number of real double-points | Maximum number of double-points | Number of imaginary double-points |
|---|---|---|---|---|---|
| $C_4^3$ | 3 | 4 | 3 | 3 | 0 |
| $C_6^5$ | 5 | 6 | 10 | 10 | 0 |
| $C_8^5$ | 5 | 8 | 15 | 21 | 6 |
| $C_8^7$ | 7 | 8 | 21 | 21 | 0 |
| $C_{10}^7$ | 7 | 10 | 28 | 36 | 8 |
| $C_{12}^7$ | 7 | 12 | 35 | 55 | 20 |
| $C_{10}^9$ | 9 | 10 | 36 | 36 | 0 |
| $C_{12}^9$ | 9 | same | as | $C_4^3$ | |
| $C_{14}^9$ | 9 | 14 | 54 | 78 | 24 |
| $C_{16}^9$ | 9 | 16 | 63 | 105 | 42 |
| $C_{12}^{11}$ | 11 | 12 | 55 | 55 | 0 |
| $C_{14}^{11}$ | 11 | 14 | 66 | 78 | 14 |
| $C_{16}^{11}$ | 11 | 16 | 77 | 105 | 28 |
| $C_{18}^{11}$ | 11 | 18 | 88 | 136 | 48 |
| $C_{20}^{11}$ | 11 | 20 | 99 | 171 | 72 |

B. $m_1$ even.

| Designation | $m_1$ | Order $=2k$ | Number of real double-points | Maxim. number of double-points | Number of imaginary double-points |
|---|---|---|---|---|---|
| $C_6^4$ | 4 | 6 | 8 | 10 | 2 |
| $C_8^6$ | 6 | same | as | $C_4^3$ | |
| $C_{10}^6$ | 6 | 10 | 24 | 36 | 12 |
| $C_{10}^8$ | 8 | 10 | 32 | 36 | 4 |
| $C_{12}^8$ | 8 | same | as | $C_6^4$ | |
| $C_{14}^8$ | 8 | 14 | 48 | 78 | 30 |
| $C_{12}^{10}$ | 10 | same | as | $C_6^5$ | |
| $C_{14}^{10}$ | 10 | 14 | 60 | 78 | 18 |
| $C_{16}^{10}$ | 10 | same | as | $C_8^5$ | |
| $C_{18}^{10}$ | 10 | 18 | 70 | 136 | 66 |
| $C_{14}^{12}$ | 12 | 14 | 72 | 78 | 6 |
| $C_{16}^{12}$ | 12 | same | as | $C_4^3$ | |
| $C_{18}^{12}$ | 12 | 18 | 96 | 136 | 40 |
| $C_{20}^{12}$ | 12 | same | as | $C_{10}^6$ | |
| $C_{22}^{12}$ | 12 | 22 | 120 | 210 | 90 |

## V. Equation of Curves of Type $C$.

From the parametric expressions for $x$ and $y$ is seen that $x$ does not change its sign by replacing $v$ by $-v$, while $y$ changes sign, so that the curve is symmetric with respect to the $x$-axis, as has been stated before. Consequently the polar equation $f(\rho, \theta)=0$ of the curve remains unchanged by replacing $\theta$ by $-\theta$. The cartesian equation is of degree $2k$, and contains only even powers of $y$, and may be written in the form

$$
\begin{aligned}
(29)\qquad & x^{2k}+a_{02}x^{2k-2}y^2+a_{04}x^{2k-4}y^4+\cdots\cdots+a_{0,2k}y^{2k}\\
&+a_{11}x^{2k-1}+a_{13}x^{2k-3}y^2+a_{15}x^{2k-5}y^4+\cdots\cdots+a_{1,2k-1}xy^{2k-2}\\
&+a_{22}x^{2k-2}+a_{24}x^{2k-4}y^2+a_{26}x^{2k-6}y^4+\cdots\cdots+a_{2,2k}y^{2k-2}\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
&+cy^2+d=0.
\end{aligned}
$$

Introducing polar coordinates $x=\rho\cos\theta$, $y=\rho\sin\theta$ this equation assumes the form

$$
\begin{aligned}
(30)\qquad & \rho^{2k}\varphi_0(\cos\theta,\sin\theta)+\rho^{2k-1}\varphi_1(\cos\theta,\sin\theta)+\rho^{2k-2}\varphi_2(\cos\theta,\sin\theta)\\
&+\cdots\cdots+\rho^{2k-1}\varphi_i(\cos\theta,\sin\theta)+\cdots\cdots+d=0.
\end{aligned}
$$

$\varphi_j(\cos\theta,\sin\theta)$, $j=0,1,2,\cdots\cdots$, are homogeneous polynomials of $\cos\theta$ and $\sin\theta$ of degree $(2k-j)$, which may be expressed as polynomials of $\cos\theta$ only, since only even powers of $y$ occur in (29). The curve, having an $m_1$-fold symmetry, has the lines

$$
y=\tan\left(\frac{2k\pi}{m_1}\right).x,\qquad k=0,1,2,3,\cdots\cdots,m_1-1,
$$

as axes of symmetry, so that the polar equation is invariant under the substitution $\left(\theta,\pm\theta+\dfrac{2k\pi}{m_1}\right)$. The only polynomials $\varphi_j$ of $\cos\theta$ which remain invariant for this substitution are of the form $c_j(\cos^2\theta+\sin^2\theta)^{n_j}$ $F(\cos m_1\theta)$, where $F$ is a polynomial of $\cos m_1\theta$. But $\cos m_1\theta$ is itself a homogeneous polynomial of $\cos\theta$ and $\sin\theta$ of degree $m_1$, and as $m_1<2k<2m_1$, the degree of $F$ can certainly not be greater than 1. $\varphi_j$ is homogeneous polynomial of degree $(2k-j)$ in $\cos\theta$ and $\sin\theta$, so that $F(\cos m_1\theta)$ reduces necessarily to $c_j\cos m_1\theta$. The $\varphi_j$'s are therefore necessarily of the form

$$
(31)\qquad \varphi_j=c_j(\cos^2\theta+\sin^2\theta)^{n_j}\cos m_1\theta
$$

of degree $(2n_j+m_1)$ in $\cos\theta$, with $2n_j+m_1\leqq 2k$. As the degree of $\varphi_j$ is $(2k-j)$, we find $n_1=(2k-j-m_1)/2$, and from this $j\leqq 2k-m_1$. For $j=0$, $c_0\varphi_0(\cos\theta,\sin\theta)=(\cos^2\theta+\sin^2\theta)^{2k-m_1}.\cos m_1\theta$. To find the intersections of the curve with the line at infinity, divide (29) through by $x^{2k}$, and solve the resulting equation for $m=y/x$, with $x\doteq\infty$, $y\doteq\infty$. The roots $m$ of the equation are the slopes of the asymptotes. This equation is equivalent with putting

$$
\lim_{x=\infty}\left\{\frac{c_0\rho^{2k}\varphi_0(\cos\theta,\sin\theta)}{x^{2k}}\right\}=(1+m^2)^{2k-m_1}\sec^{m_1}\theta.\cos m_1\theta=0.
$$

The roots of this equation are $m=+j$, $m=-j$, each $(2k-m_1)$-fold, and $m_1\theta=(2\nu_j+1)\dfrac{\pi}{2}$, $\nu_j=1, 2, 3, \cdots\cdots$. This shows that under the assumption of $\varphi_0$ in the form (31) the curve would have real infinite points whose directions are given by the angles $\theta=(2\nu_1+1)\,\pi/2m_1$, which is not possible. We have therefore necessarily $n_0=2k$, as the degree of $\varphi_0$, so that the first term of (30) is simply $c_0\rho^{2k}$, and the curve has at each circular point $k$ points of intersection with the line at infinity.

## Case when $m_1$ is odd.

From the possible forms of $\varphi_j$ in (31) follows that in this case the polar equation of the curve may be written in the form

$$(32)\qquad \rho^{2k}+a_1\rho^{2k-2}+a_2\rho^{2k-4}+\cdots\cdots+a_j\rho^2+a$$
$$+(b_1\rho^{2\lambda_1}+b_2\rho^{2\lambda_2}+\cdots\cdots+b_j\rho^2+b)\,\rho^{m_1}\cos m_1\,\theta=0,$$

where $a\neq 0$, $\lambda_1>\lambda_2>\lambda_3>\cdots\cdots>1$, and $2\lambda_1+m_1\leqq 2k-1$. When $m_1$ is odd,

$$\cos m_1\,\theta=\cos^{m_1}\theta-\binom{m_1}{2}\cos^{m_1-2}\theta\cdot\sin^2\theta+\binom{m_1}{2}\cos^{m_1-4}\theta\cdot\sin^4\theta$$
$$+\cdots\cdots+(-1)^{\frac{m_1-1}{2}}\binom{m_1}{m_1-1}\cos\theta\cdot\sin^{m_1-1}\theta.$$

Introducing again cartesian coordinates (32) may be written as

$$(33)\quad (x^2+y^2)^k+a_1(x^2+y^2)^{k-1}+a_2(x^2+y^2)^{k-2}+\cdots\cdots a_j(x^2+y^2)+a$$
$$+\{b_1(x^2+y^2)^{\lambda_1}+b_2(x^2+y^2)^{\lambda_2}+\cdots\cdots+b_j(x^2+y^2)+b\}.$$
$$\Big\{x^{m_1}-\binom{m_1}{2}x^{m-2}y^2+\binom{m_1}{4}x^{m-4}y^4$$
$$+\cdots\cdots+(-1)^{\frac{m_1-1}{2}}xy^{m_1-1}\Big\}=0.$$

In order to investigate the character of the singularity at the circular points we must first make an involutory perspective of the curve, $x=x'/(y'-1)$, $y=y'/(y'-1)$, by which the circular points at infinity are transformed into the points $I'(-i, 1)$, $J'(+i, 1)$. Next we make the transformation $x'^2+y'^2=x''$, $y'-1=y''$, which, as can easily be established, does not change the character of a singularity at either point $I'$ or $J'$, nor any other finite point, except $(x'=0, y'=0)$. The product of the two transformations is equivalent with

$$(34) \qquad x=\pm\sqrt{x''-y''^2-2y''-1}\,/y'', \quad y=(y''+1)/y'',$$

where the $+$sign has to be taken for the circular point with the slopeo f its direction equal to $-i$, the $-$sign for the slope $+i$. As $x^2+y^2=x''/y''^2$, equation (33) after the transformation, and after suppressing the double-primes, assumes the form

$$\begin{aligned} & x^k+a_1x^{k-1}y^2+a_2x^{k-2}y^4+\cdots\cdots+a_jxy^{2k-2}+ay^{2k} \\ & \quad +\{b_1x^{\lambda_1}.\,y^{2k-2\lambda_1}+b_2x^{\lambda_2}\,y^{2k-2\lambda_2}+\cdots\cdots+b_jx.\,y^{2k-2}+by^{2k}\} \\ & \quad \times\Big[\pm\sqrt{x-y^2-2y-1}\,\Big\{(x-y^2-2y-1)^{\frac{m_1-1}{2}} \\ & \quad -\binom{m_1}{2}(x-y^2-2y-1)^{\frac{m_1-3}{2}}.(y+1)^2+\cdots\cdots\Big\}\,y^{-m_1}\Big]=0. \end{aligned}$$

Getting rid of the radical by squaring we get from this

$$x^{2k}+\text{higher terms}=(x-y^2-2y-1)\,(\alpha+\beta\,x+\gamma\,y+\cdots\cdots)y^{4k-2m_1},$$

or finally

$$(35) \qquad x^{2k}+\text{higher terms}=\alpha\,y^{4k-2m_1}+\text{higher terms}.$$

"Placing" this equation on the analytic triangle(1), we find that the curve has the same singularities at the point $(x=0,\ y=0)$ as the binomial curve $y^{4k-2m_1}=x^{2k}$.

But to $x=0$, $y=0$, according to the $+$, or $-$ sign in (34) correspond in the original curve (33) the circular points at infinity. The singularity at a circular point has therefore the same character as the binomial curve

$$(36) \qquad \sqrt{\alpha}\,.y^{2k-m_1}=x^k$$

at the origin This must therefore be counted for $(k-1)\,(2k-m_1-1)\,/\,2$ double points(2), so that the circular points at infinity absorb $(k-1)\times(2k-m_1-1)$ double-points. Now the number of real double-points is $m_1(k-1)$, so that the total number of double-points of the curve is

$$m_1(k-1)+(k-1)\,(2k-m_1-1)=\frac{(2k-2)\,(2k-1)}{2},$$

---

(1) Wieleitner: *Theorie der ebenen algebraischen Kurven höherer Ordnung*, pp. 83–136.

(2) Ch Scott: *On the Higher Singularities of Plane Curves*, American Journal of Mathematics, Vol. 14, pp. 301–325.

i. e., the maximum which a curve of order $2k$ may have. We have therefore verified directly from the equation of the curve, that it is rational, a fact established before by an entirely different method. When the curve has all double-points real, then its degree for $m_1$ odd is $m_1+1=2k$, so that from the preceding formula the number of imaginary double points is clearly zero, which is another verification of previous results.

## Case when $m_1$ is even.

In this case the equation of the curve contains even powers of $x$ and $y$ only; the curve is symmetric with respect to the $x-$and $y-$axis. The application of the transformation (34) to the curve is even simpler than in the case, where $m_1$ is odd, and leads to precisely the same results.

A circle with the centre at the origin cuts the curve in $2m_1$ finite points, so that each circular point must be counted as a $(2k-m_1)-$fold point of intersection of the circle with the curve. To verify this directly, we find the number of intersections of the curves

$$\sqrt{a}\,.y^{2k-m_1}=x^k, \text{ and } x=r\,y^2,$$

the latter being the transformed by (34) of the circle $x^2+y^2=r^2$. The simultaneous solutions in $y$ are given by

$$y^{2k-m_1}.(\sqrt{a}-r^k y^{m_1})=0.$$

This shows clearly that there are $2k-m_1$ intersections at the origin, and no more, and consequently $2k-m_1$ intersections at each circular point of the circle and the original curve.

## VI. Examples.

### 1. $C_4^3$, $m_1=3$.

The polar equation of this curve may be written in the form

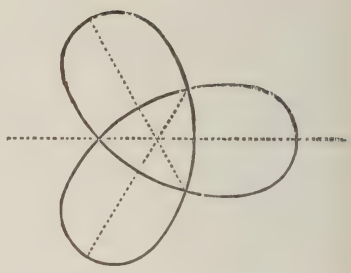
$$\rho^4-R\,\rho^3\cos 3\theta-\frac{27}{4}\,R^2\,\rho^2+\frac{27}{4}\,R^4=0.$$

---

(1) G. de Longchamps: *Journal de Mathématiques Élémentaires*, 2e série, Vol. IV, pp. 269–277 (1885)

Also H. Brocard: *Journal de Mathématiques Spéciales*, 3. series, Vol. V, pp. 56–64 (1891).

It intersects the $x$-axis in the two single points $(R, 0)$, $(3R, 0)$, and the double point $(-\frac{3R}{2}, 0)$, Fig. 1. When $R=\frac{1}{4}$, its rational parametric expression is



$$x=\frac{t^4-12t^2+3}{4(1+t^2)^2}, \quad y=\frac{2t(3-5t^2)}{4(1+t^2)^2}.$$

**2.** $C_6^4$, $m_1=4$.

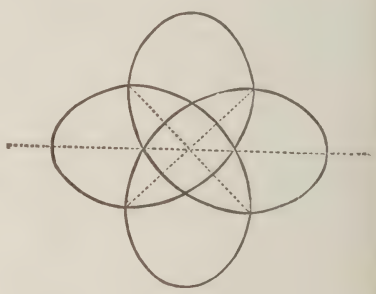
This is a bicircular sextic with 8 real double points. The cartesian equation of a curve of this type may be written in the form

$$3(x^2+y^2)^3-24(x^4+y^4)-32x^2y^2+39(x^2+y^2)-18=0.$$

The polar equation is

$$3\rho^6-(2\cos 4\theta+22)\rho^4+39\rho^2-18=0.$$

The distances of the two double points from the origin, on each of the $x$-and $y$-axis, are equal to 1, the distances of the single points equal to $\sqrt{6}$. On the bisectors of the four quadrants the distances of the double points from the origin are equal to $\sqrt{3}$, those of the single points $\sqrt{\frac{2}{3}}$, Fig. 2.

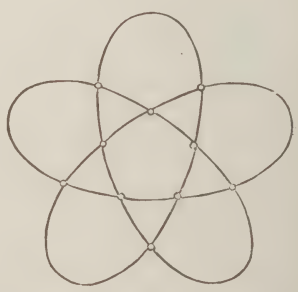


**3.** $C_6^5$, $m_1=5$.

This is a sextic with 10 (all) real double points, Fig. 3.

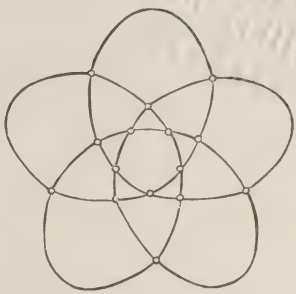
**4.** $C_8^5$, $m_1=5$.

This is a curve of order 8, with 15 real double points and triple-point at each circular point at infinity, Fig. 4.



The figures for the curves $C_6^5$ and $C_8^5$ have been drawn upon the basis of

their topological properties and must be considered merely as sketches, and not as graphs of their algebraic equations. The equations of $C_4^3$ and $C_6^4$ have been established upon the basis of their geometric properties, and not from their parametric representations with the time as a parameter. The establishment of the cartesian equations of these curves from $x$ and $y$ as functions of the time is connected with considerable difficulties.



Mechanical means to generate and graph spherical pendulum curves were devised by Professor A. G. Webster(1). These curves, however, are not closed.

(1) *The Dynamics of Particles and of Rigid, Elastic, and Fluid Bodies*, pp. 48-55.

# On a Transformation Theorem relating to Spheroidal Harmonics,

by

BIBHUTIBHUSON DATTA, Calcutta, India.

The object of this paper is to establish certain theorems by means of which an external spheroidal harmonics corresponding to a spheroid $S_2$ can be expressed in terms of the spheroidal harmonics corresponding to another spheroid $S_1$ having its centre different from that of $S_2$. It is believed that the theorems $(B)$ and $(C)$ are new, as none of the previous writers on the subject, including Sir W. D. Niven and Mr. G. B. Jeffery, seem to have noticed them.

I take $x_2, y_2, z_2$ to be the rectangular cartesian coordinates of a point when referred to $O_2$, the centre of $S_2$, as origin, and $x_1, y_1, z_1$ to be the coordinates of the same point when referred to $O_1$, the centre of $S_1$, as origin; it being understood that the axes of revolution of $S_2$ and $S_1$ are respectively along the $z_2$ and $z_1$ axes.

**1.** Let $\mathfrak{G}_n^6(2)$ denote external spheroidal harmonics referred to the second system of coordinates. Then if $a_2$ and $c_2$ be the semiaxes of the spheroid, we have

$$\mathfrak{G}_n^6(2)=(-1)^n \frac{2^{n+1}\, n!}{(2n+1)!} H_n^6\left(\frac{\partial}{\partial x_2}, \frac{\partial}{\partial y_2}, \frac{\partial}{\partial z_2}\right) \cdot \left[1+\frac{k_2^2}{2(2n+3)} \frac{\partial^2}{\partial z_2^2}+\cdots\cdots\right] \frac{1}{r_2} \text{ (1)},$$

where

$$r_2^2=x_2^2+y_2^2+z_2^2 \quad \text{and} \quad k_2^2=-a_2^2+c_2^2.$$

Again

(1) Vide W. D. Niven, "On Ellipsoidal Harmonics" *Phil. Trans*, Vol. 182 (A) p. 245; Hobson, "On the evaluation of a certain surface integral and its application to the expansion in series of the Potential of Ellipsoids," *Proc. Lond. Math. Soc.*, Vol. XXIV, p. 91.

$$H_n^\sigma\left(\frac{\partial}{\partial x_2}, \frac{\partial}{\partial y_2}, \frac{\partial}{\partial z_2}\right)\frac{1}{r_2}=(-1)^n\frac{(2n)!}{2^n\,n!}\,\frac{H_n^\sigma(x_2, y_2, z_2)}{r_2^{2n+1}}\ (^1),$$

and

$$H_n^\sigma(x_2, y_2, z_2)=\frac{(n-\sigma)!}{2^{\sigma-1}\,n!}\,r_2^n\,P_n^\sigma(\mu_2)\cos\sigma\phi_2\ (^2),$$

where

$$P_n^\sigma(\mu_2)=(1-\mu_2)^{\frac{\sigma}{n}}\frac{d^\sigma}{d\mu_2^\sigma}P_n(\mu_2).$$

Also

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{\cos\sigma u\,du}{(z_2+ix_2\cos u+iy_2\sin u)^{n+1}}=(-i)^\sigma\frac{(n-\sigma)!}{n!}\,\frac{P_n^\sigma(\mu_2)}{r_2^{n+1}}\cos\sigma\phi_2\ (^3).$$

Therefore

$$H_n^\sigma\left(\frac{\partial}{\partial x_2}, \frac{\partial}{\partial y_2}, \frac{\partial}{\partial z_2}\right)\frac{1}{r_2}=(-1)^n\frac{i^\sigma(2n)!}{2^{n+\sigma}n!}\,\frac{1}{\pi}\int_{-\pi}^{\pi}\frac{\cos\sigma u}{\zeta_2^{n+1}}\,du,$$

where $\zeta_2$ denotes

$$z_2+ix_2\cos u+iy_2\sin u.$$

Hence

$$\mathfrak{G}_n^\sigma(2)=\frac{i^\sigma}{2^{\sigma-1}(2n+1)}\,\frac{1}{\pi}\int_{-\pi}^{\pi}\left[\frac{1}{\zeta_2^{n+1}}+\frac{(n+1)(n+2)}{2(2n+3)}\,\frac{k_2^2}{\zeta_2^{n+3}}+\cdots\right]\cos\sigma u\,du$$

$$=i^\sigma\frac{(2n)!}{2^{n+\sigma-1}\,n!\,n!}\,\frac{1}{k_2^{n+1}}\,\frac{1}{\pi}\int_{-\pi}^{\pi}Q_n\left(\frac{\zeta_2}{k_2}\right)\cos\sigma u\,du.$$

**2.** Now

$$\frac{1}{(z_2+ix_2\cos u+iy_2\sin u)^{n+1}}=\frac{1}{(s-z_1+ix_1\cos u+iy_1\sin u)^{n+1}}$$

$$=\frac{1}{k_1^{n+1}}\,\frac{1}{\left\{\dfrac{s}{k_1}-\dfrac{x_1\cos u+y_1\sin u+iz_1}{ik_1}\right\}^{n+1}}.$$

If $\left|\dfrac{s}{k_1}\right|>\left|\dfrac{x_1\cos u+y_1\sin u+iz_1}{ik_1}\right|$, the right hand side of this equation can be expanded in a series of Legendre's polynomials. Thus

(1) Niven, p. 236.
(2) *Ibid*, p. 261.
(3) Whittaker and Watson, "Modern Analysis"

$$\frac{1}{\zeta_2^{n+1}}=\frac{(-1)^n}{n!}\frac{1}{k_1^{n+1}}\sum_{m=0}^{m=\infty}(2m+1)\,P_m\left(\frac{\zeta_1'}{i\,k_1}\right)\frac{d^n}{dt^n}\,Q_m(t)\,(^1),$$

where

$$\zeta_1'=x_1\cos u+y_1\sin u+iz_1,\quad\text{and}\quad t=s/k_1.$$

Therefore

$$\mathfrak{G}_n^\sigma(2)=(-1)^n\frac{i^\sigma}{2^{\sigma-1}(2n+1)n!}\frac{1}{k_1^{n+1}}\sum_{m=0}^{\infty}(2m+1)\frac{1}{\pi}\int_{-\pi}^{\pi}P_m\left(\frac{\zeta_1'}{ik_1}\right)\cos\sigma u\,du$$

$$\times\left[D_t^n+\frac{\rho^2}{2(2n+3)}D_t^{n+2}+\cdots\cdots\right]Q_m(t),$$

where $D_t$ denotes $\dfrac{d}{dt}$, and $\rho=k_2/k_1$. Or

$$\frac{(2n)!}{2^n\,n!}\cdot\frac{1}{k_2^{n+1}}\cdot\frac{1}{\pi}\int_{-\pi}^{\pi}Q_n\left(\frac{\zeta_2}{k_2}\right)\cos\sigma u\,du$$

$$=(-1)^n\frac{1}{(2n+1)}\frac{1}{k_1^{n+1}}\sum_{m=0}^{\infty}(2m+1)\frac{1}{\pi}\int_{-\pi}^{\pi}P_m\left(\frac{\zeta_1'}{ik_1}\right)\cos\sigma u\,du$$

$$\times\left[D_t^n+\frac{\rho^2}{2(2n+3)}D_t^{n+2}+\cdots\cdots\right]Q_m(t).\qquad\text{(A)}$$

**3.** We shall now express this in terms of spheroidal coordinates. Supposing the spheroids are *prolate*, we write

$$x_1=k_1\sin\theta_1\sinh\eta_1\cos\omega_1=k_1(1-\mu_1^2)^{\frac{1}{2}}(\lambda_1^2-1)^{\frac{1}{2}}\cos\omega_1,$$

$$y_1=k_1\sin\theta_1\sinh\eta_1\sin\omega_1=k_1(1-\mu_1^2)^{\frac{1}{2}}(\lambda_1^2-1)^{\frac{1}{2}}\sin\omega_1,$$

$$z_1=k_1\cos\theta_1\cosh\eta_1=k_1\,\mu_1\,\lambda_1.$$

Then

$$P_m\left(\frac{\zeta_1'}{ik_1}\right)=P_m\{\mu_1\,\lambda_1-i(1-\mu_1^2)^{\frac{1}{2}}(\lambda_1^2-1)^{\frac{1}{2}}\cos(u-\omega_1)\}$$

$$=P_m\{\mu_1\,\lambda_1+(1-\mu_1^2)^{\frac{1}{2}}(1-\lambda_1^2)^{\frac{1}{2}}\cos(u-\omega_1)\}$$

$$=P_m(\mu_1)\,P_m(\lambda_1)$$

$$+2\sum_{q=1}^{q=m}\frac{(m-q)!}{(m+q)!}P_m^q(\mu_1)\,P_m^q(\lambda_1)\cos q(u-\omega_1).$$

Therefore

(1) Todhunter, "The Functions of Laplace, Lamé and Bessel," p. 88.

$$\int_{-\pi}^{\pi} P_m\left(\frac{\zeta_1'}{ik_1}\right)\cos\sigma u\,du = 2\pi\,\frac{(m-\sigma)!}{(m+\sigma)!}\,P_m^\sigma(\mu_1)\,P_m^\sigma(\lambda_1)\cos\sigma\omega_1\,.$$

Similarly

$$\int_{-\pi}^{\pi} Q_n\left(\frac{\zeta_2}{k_2}\right)\cos\sigma u\,du = 2\pi\,\frac{(n-\sigma)!}{(n+\sigma)!}\,P_n^\sigma(\mu_2)\,Q_n^\sigma(\lambda_2)\cos\sigma\omega_2\,.$$

Substituting in (A) we get

$$\frac{(n-\sigma)!}{(n+\sigma)!}\,P_n^\sigma(\mu_2)\,Q_n^\sigma(\lambda_2)\cos\sigma\omega_2$$

$$=\sum_{m=\sigma}^{\infty}(2m+1)\,\frac{(m-\sigma)!}{(m+\sigma)!}\,\varpi_1(m,n)\,P_m^\sigma(\mu_1)\,P_m^\sigma(\lambda_1)\cos\sigma\omega_1\,, \quad \text{(B)}$$

where

$$\varpi_1(m,n)=(-1)^n\,\frac{2^n\,n!}{(2n+1)!}\,\rho^{n+1}\left[D_t^n+\frac{\rho^2}{2(2n+3)}D_t^{n+2}+\cdots\cdots\right]Q_m(t).$$

**4.** When the spheroids are of the *oblate* or planetary form, the appropriate coordinates are given by

$$x_1=k_1'\sin\theta_1\cosh\eta_1\cos\omega_1=k_1'(1-\mu_1^2)^{\frac{1}{2}}(\lambda_1^2+1)^{\frac{1}{2}}\cos\omega_1\,,$$

$$y_1=k_1'\sin\theta_1\cosh\eta_1\sin\omega_1=k_1'(1-\mu_1^2)^{\frac{1}{2}}(\lambda_1^2+1)^{\frac{1}{2}}\sin\omega_1\,,$$

$$z_1=k_1'\cos\theta_1\sinh\eta_1=k_1'\,\mu_1\,\lambda_1\,.$$

Putting $k_1'=ik_1$ and proceeding exactly in the same way, we get, for oblate spheroids,

$$\frac{(n-\sigma)!}{(n+\sigma)!}\,P_n^\sigma(\mu_2)\,Q_n^\sigma(i\lambda_2)\cos\sigma\omega_2$$

$$=(-1)^\sigma\sum_{m=\sigma}^{\infty}(2m+1)\,\frac{(m-\sigma)!}{(m+\sigma)!}\,\varpi_1(m,n)\,P_m^\sigma(\mu_1)\,P_m^\sigma(i\lambda_2)\cos\sigma\omega_1\,. \quad \text{(C)}$$

**5.** When $a=c$, so that the spheroids become spheres, we have $k_1=0$, and $k_2=0$.

Then

$$\frac{1}{k_2^{n+1}}\cdot\frac{1}{\pi}\int_{-\pi}^{\pi}Q_n\left(\frac{\zeta_2}{k_2}\right)\cos\sigma u\,du=\frac{2^n\,n!\,n!}{(2n+1)!}\,\frac{1}{\pi}\int_{-\pi}^{\pi}\frac{\cos\sigma u}{\zeta_2^{n+1}}\,du$$

$$=(-i)^\sigma\,\frac{2^{n+1}\,n!\,(n-\sigma)!}{(2n+1)!}\,\frac{P_n^\sigma(\mu_2)}{r_2^{n+1}}\cos\sigma\phi_2\,.$$

$$\frac{k_1^m}{\pi}\int_{-\pi}^{\pi} P_m\left(\frac{\zeta_1'}{ik_1}\right)\cos\sigma u\,du=\frac{i^\sigma(2m)!}{2^{m-1}m!\,(m+\sigma)!}\,r_1^m\,P_m^\sigma(\mu_1)\cos\sigma\phi_1\,.$$

Further

$$\frac{1}{k_1^{m+n+1}}D_t^n Q_m(t)=\frac{1}{k_1^{m+1}}D_s^n Q_m\left(\frac{s}{k_1}\right)=(-1)^n\frac{2^m m!\,(m+n)!}{(2m+1)!}\,\frac{1}{s^{m+n+1}}\,.$$

Hence the equation (A) becomes

$$\frac{P_n^\sigma(\mu_2)}{r_2^{n+1}}\cos\sigma\phi_2=(-1)^\sigma\sum_{m=\sigma}^{\infty}\frac{(m+n)!}{(n-\sigma)!\,(m+\sigma)!}\,r_1^m\,P_m^\sigma(\mu_1)\cos\sigma\phi_1\,.$$

**6.** If

$$\left|\frac{s}{k_1}\right|<\left|\frac{x_1\cos u+y_1\sin u+iz_1}{ik_1}\right|$$

we have the expansion

$$\frac{1}{\left\{\dfrac{s}{k_1}-\dfrac{x_1\cos u+y_1\sin u+iz_1}{ik_1}\right\}^{n+1}}$$

$$=\frac{(-1)^{n+1}}{n!}\,\frac{1}{k_1^{n+1}}\sum_{m=n}^{\infty}(2m+1)\,Q_m\left(\frac{\zeta'}{ik_1}\right)\frac{d^n}{dt^n}P_m(t):$$

for as we know if $m<n$, $\dfrac{d^n}{dt^n}P_m(t)=0$.

Therefore

$$\mathfrak{G}_n^\sigma(2)=(-1)^{n+1}\frac{i^\sigma}{2^{\sigma-1}(2n+1)n!}\cdot\frac{1}{k_1^{n+1}}\sum_{m=n}^{\infty}(2m+1)\frac{1}{\pi}\int_{-\pi}^{\pi}Q_m\left(\frac{\zeta'}{ik_1}\right)\cos\sigma u\,du$$

$$\times\left[D_t^n+\frac{\rho^2}{2(2n+3)}D_t^{n+2}+\cdots\cdots\right]P_m(t)\,;$$

and corresponding to (A) we have now

$$\frac{(2n)!}{2^n n!}\cdot\frac{1}{k_2^{n+1}}\,\frac{1}{\pi}\int_{-\pi}^{\pi}Q_n\left(\frac{\zeta_2}{k_2}\right)\cos\sigma u\,du$$

$$=\frac{(-1)^{n+1}}{(2n+1)}\cdot\frac{1}{k_1^{n+1}}\sum_{m=n}^{\infty}(2m+1)\,\frac{1}{\pi}\int_{-\pi}^{\pi}Q_m\left(\frac{\zeta'}{ik_1}\right)\cos\sigma u\,du$$

$$\times\left[D_t^n+\frac{\rho^2}{2(2n+3)}D_t^{n+2}+\cdots\cdots\right]P_m(t)\,. \qquad \text{(A}'\text{)}$$

Transforming to spheroidal coordinates we get for *prolate* spheroids

$$\frac{(n-\sigma)!}{(n+\sigma)!} P_n^\sigma(\mu_2)\, Q_n^\sigma(\lambda_2) \cos\sigma\omega_2$$

$$= \sum_{m=n}^{\infty} (2m+1)\, \varpi_1'(m, n) \frac{(m-\sigma)!}{(m+\sigma)!} P_m^\sigma(\mu_1)\, Q_m^\sigma(\lambda_1) \cos\sigma\omega_1\,, \qquad \text{(D)}$$

where

$$\varpi_1'(m, n) = (-1)^{n+1} \frac{2^n\, n!}{(2n+1)!} \rho^{n+1} \left[ D_t^n + \frac{\rho^2}{2(2n+3)} D_t^{n+2} + \cdots\cdots \right] P_m(t);$$

and for *oblate* spheroids

$$\frac{(n-\sigma)!}{(n+\sigma)!} P_n^\sigma(\mu_2)\, Q_n^\sigma(i\lambda_2) \cos\sigma\omega_2$$

$$= (-1)^\sigma \sum_{m=n}^{\infty} (2m+1)\, \varpi_1'(m, n)\, P_m^\sigma(\mu_1)\, Q_m^\sigma(i\lambda_1) \cos\sigma\omega_1\,. \qquad \text{(E)}$$

# On Certain Mean Curves defined by the Series of Orthogonal Functions,

by

KINNOSUKE OGURA, Ôsaka.

**1.** Let

$$\varphi_0(t),\ \varphi_1(t),\ \varphi_2(t),\cdots\cdots,\varphi_n(t),\cdots\cdots$$

be a system of orthogonal functions which are *limited* and *integrable* in the interval $(0 \leqq t \leqq 1)$ and are such that

$$\int_0^1 \varphi_m(t)\,\varphi_n(t)\,dt = 0 \quad (m \neq n),$$
$$= 1 \quad (m = n), \quad (m, n = 0, 1, 2, \cdots\cdots).$$

Let $f(\theta)$ be a function which is *limited* and *integrable* in the interval $(0 \leqq \theta \leqq 1)$, then the series of orthogonal functions $\varphi_n(\theta)$ corresponding to $f(\theta)$ is

$$(1) \qquad \sum_{n=0}^{\infty} \varphi_n(\theta) \int_0^1 f(t)\,\varphi_n(t)\,dt\,.$$

Now consider the plane curve $C$ defined by

$$(2) \qquad x = x(\theta), \qquad y = y(\theta),$$

where the two functions $x(\theta)$ and $y(\theta)$ are *limited* and *integrable* in the interval $(0 \leqq \theta \leqq 1)$; and denote by $D$ *the closed least convex domain which contains the curve* $C$.

If we put

$$(3) \qquad \begin{aligned} x_n^{(r)}(\theta) &= \int_0^1 x(t)\,K_n^{(r)}(\theta, t)\,dt, \\ y_n^{(r)}(\theta) &= \int_0^1 y(t)\,K_n^{(r)}(\theta, t)\,dt, \end{aligned}$$

where

$$(4)\quad K_n^{(r)}(\theta, t)$$
$$=\varphi_0(\theta)\varphi_0(t)+\frac{n-1}{n+r-1}\varphi_1(\theta)\varphi_1(t)+\frac{(n-1)(n-2)}{(n+r-1)(n+r-2)}\varphi_2(\theta)\varphi_2(t)$$
$$+\cdots\cdots+\frac{(n-1)!}{(n+r-1)(n+r-2)\cdots(r+1)}\varphi_{n-1}(\theta)\varphi_{n-1}(t),$$

the curve $C_n^{(r)}$ defined by

$$(5)\qquad x=x_n^{(r)}(\theta),\quad y=y_n^{(r)}(\theta)$$

will be called *the $n^{th}$ curve of the $r^{th}$ mean (in Cesàro's sense) of $C$.*

Prof. Fejér proved that when

$$1,\ \sqrt{2}\cos 2\pi t,\ \sqrt{2}\sin 2\pi t,\cdots\cdots,\ \sqrt{2}\cos 2n\pi t,\ \sqrt{2}\sin 2n\pi t,\cdots\cdots$$

are taken as the system of orthogonal functions, then *all* the curves $C_n^{(1)}(n=1, 2,\cdots\cdots)$ of *any* curve $C$ are contained in the domain $D$ (1).

In this note I propose to find *a necessary and sufficient condition that all the curves $C_n^{(r)}$ $(n=1, 2,\cdots\cdots)$ of any curve $C$ should be contained in the domain $D$* (§ 4). Also I will give a theorem on the *summability* (the convergency of the sequence of $C_n^{(r)}$ $(n=1, 2,\cdots\cdots)$) in § 6, and lastly in § 7 remark that *all the sum curves $C_n^{(0)}$ $(n=1, 2,\cdots\cdots)$ of any curves $C$ for the series of Haar's functions are contained in $D$.*

## I.

**2.** I will begin with the *lemma*:

Let $K(t)$ be a given function which is limited and integrable in the interval $(0\leqq t\leqq 1)$. Then in order that the point

$$X=\int_0^1 x(t)\,K(t)\,dt,\qquad Y=\int_0^1 y(t)\,K(t)\,dt$$

may be contained in the domain $D$ for *any* curve $C$

$$x=x(t),\qquad y=y(t),$$

it should be

$$K(t)\geqq 0$$

at every point of the interval $(0\leqq t\leqq 1)$ at which it is continuous.

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(1) Fejér, "Über gewisse durch die *Fourier*sche und *Laplace*sche Reihe definierten Mittelkurven und Mittelflächen," Rend. Palermo, 38 (1914), p. 79.

Since $K(t)$ is limited and integrable in the interval $(0 \leqq t \leqq 1)$, it must have points of continuity in any subinterval of $(0 \leqq t \leqq 1)$. Let us denote by $(A)$ all the subintervals in which $K(t) \geqq 0$ and by $(B)$ those in which $K(t) < 0$. If we take

$$\begin{aligned} x(t) &= 1 \qquad \text{when } t \text{ belongs to } (B), \\ &= 0 \qquad \text{when } t \text{ does not belong to } (B); \\ y(t) &= \sin 2\pi t \text{ throughout } 0 \leqq t \leqq 1, \end{aligned}$$

then the domain $D$ for the curve $C$ is enclosed in the rectangle

$$0 \leqq x \leqq 1, \qquad -1 \leqq y \leqq 1.$$

But since

$$X = \int_{(B)} x(t)\, K(t)\, dt = \int_{(B)} K(t)\, dt < 0\,,$$

the point $(X, Y)$ lies outside the domain $D$, which proves the lemma.

Applying this lemma to (3) we have *the first necessary condition:* In order that *all* the curves $C_n^{(r)}$ $(n=1, 2, \cdots\cdots)$ of *any* curve $C$ should be contained in $D$, we must have

$$(6) \qquad K_n^{(r)}(\theta, t) \geqq 0, \quad (n=1, 2, \cdots\cdots)$$

at every point of the domain $(0 \leqq \theta \leqq 1,\ 0 \leqq t \leqq 1)$ at which they are continuous.

**3.** Next I will prove *the second lemma:*

In order that the curve $C_1^{(r)}$ of *any* curve $C$ may be contained in $D$, it is necessary that

$$\varphi_0(t) = 1 \quad \text{or} \quad -1$$

at every point of the interval $(0 \leqq t \leqq 1)$ at which it is continuous.

(i) Firstly we have from (6)

$$\varphi_0(\theta)\, \varphi_0(t) \geqq 0$$

at every point of the domain $(0 \leqq \theta \leqq 1,\ 0 \leqq t \leqq 1)$ at which $\varphi_0(\theta)\, \varphi_0(t)$ is continuous; so that $\varphi_0(t)$ must have a definite sign at all points of the interval $(0 \leqq t \leqq 1)$ at which it is continuous. (For the sake of brevity, this set of points will be donoted by $E$).

Next since

$$\int_0^1 \varphi_0^2(t)\, dt = 1\,,$$

$\varphi_0(t)$ can not vanish identically throughout $E$. So we may suppose that this function is non-negative throughout $E$.

Also from the last identity we can infer that $\varphi_0(t)$ is neither always greater nor always smaller than 1 throughout $E$.

(ii) Now let us denote by $M$ the absolute maximum of $\varphi_0(t)$ in $E$, and suppose

$$\varphi_0(\theta_1)=M>1\,,$$

where $\theta_1$ belongs to $E$. Since $\varphi_0(t)$ can not be equal to $M$ identically throughout $E$, we have

$$\int_0^1 M\,\varphi_0(t)\,dt>\int_0^1 \varphi_0^2(t)\,dt=1\,.$$

If we take the curve

$$\begin{aligned} x=x(t)&=t \qquad (0\leqq t\leqq 1);\\ y=y(t)&=0 \qquad\quad (t=0),\\ &=M \quad (0<t\leqq 1), \end{aligned}$$

the domain $D$ becomes the triangle having the vertices $(0,0)$, $(0,M)$, $(1,M)$. On the other hand, since

$$y_1^{(r)}(\theta)=\varphi_0(\theta)\int_0^1 M\,\varphi_0(t)\,dt>\varphi_0(\theta)\,,$$

we have

$$y_1^{(r)}(\theta_1)>\varphi_0(\theta_1)=M\,;$$

so that the domain $D$ can not contain the whole part of the curve $C_1^{(r)}$.

Consequently we must have $M=1$, from which it follows that $\varphi_0(t)=1$ throughout $E$.

This lemma leads us to *the second necessary condition:* In order that *all* the curves $C_n^{(r)}$ $(n=1,2,\cdots\cdots)$ of *any* curve $C$ should be contained in $D$, it must be

$$(7)\qquad \varphi_0(t)=1 \quad\text{or}\quad -1$$

at every point of the interval $(0\leqq t\leqq 1)$ at which it is continuous([1]).

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([1]) By kind information of Prof. Fujiwara we add here the following remark: It may be directly proved in order that the point $(X, Y)$ for any curve $C$ should be contained in $D$ it is necessary and sufficient that

**4.** Thus we have had two necessary conditions. Here I will prove that *these two conditions are sufficient.*

By the second condition we have

$$\varphi_0(t)=1 \quad \text{or} \quad -1$$

at every point of the interval $(0 \leqq t \leqq 1)$ at which it is continuous, so that

$$\int_0^1 \varphi_n(t)\,dt = \pm\int_0^1 \varphi_0(t)\,\varphi_n(t)\,dt$$
$$=0, \qquad (n=1, 2, \cdots\cdots).$$

But since (4) gives

$$K_n^{(r)}(\theta, t)=1+\frac{n-1}{n+r-1}\varphi_1(\theta)\varphi_1(t)+\cdots+\frac{(n-1)!}{(n+r-1)\cdots(r+1)}\varphi_{n-1}(\theta)\,\varphi_{n-1}(t),$$

we obtain *the identities:*

$$(8) \qquad \int_0^1 K_n^{(r)}(\theta, t)\,dt=1, \qquad (n=1, 2, \cdots\cdots).$$

Therefore equations (3) may be written

$$x_n^{(r)}(\theta)=\frac{\int_0^1 x(t)\,K_n^{(r)}(\theta, t)\,dt}{\int_0^1 K_n^{(r)}(\theta, t)\,dt},$$

$$(n=1, 2, \cdots\cdots),$$

$$y_n^{(r)}(\theta)=\frac{\int_0^1 y(t)\,K_n^{(r)}(\theta, t)\,dt}{\int_0^1 K_n^{(r)}(\theta, t)\,dt}.$$

But by the first condition we have

$$(6) \qquad K_n^{(r)}(\theta, t) \geqq 0 \qquad (n=1, 2, \cdots\cdots)$$

at every point of the domain $(0 \leqq \theta \leqq 1,\ 0 \leqq t \leqq 1)$ at which they are

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$$1^\circ \quad K(t) \geqq 0; \quad 2^\circ \int_0^1 K(t)\,dt=1.$$

For the present case the condition 2° is equivalent to $\varphi_0^2=1$; it is, however, proved in the text that 2° is a consequence of 1°.

continuous; consequently, by a theorem due to Weierstrass, all the curves $C_n^{(r)}$ $(n=1,2,\cdots\cdots)$ lie in the domain $D$.

Therefore we arrive at the theorem:

Theorem I. *A necessary and sufficient condition that all the curves* $C_n^{(r)}$ $(n=1,2,\cdots\cdots)$ *for any curve* $C$ *may lie in the domain* $D$ *is that the orthogonal functions* $\varphi_n(t)$ *should satisfy*

$$\varphi_0(t)=\pm 1,$$

$$K_n^{(r)}(\theta,t)\geqq 0, \qquad (n=2,3,\cdots\cdots)(^1)$$

*at every point of the domain* $(0\leqq\theta\leqq 1,\ 0\leqq t\leqq 1)$ at which $\varphi_0(t)$ and $K_n^{(r)}(\theta,t)$ *are continuous.*

## II.

**5.** As the first application of the above theorem we consider the relation between the curves $C_n^{(r)}$ and $C_n^{(r+1)}$.

By means of the formula, which is easily seen,

$$K_n^{(r+1)}(\theta,t)=\frac{A_1^{(r)}K_1^{(r)}(\theta,t)+A_2^{(r)}K_2^{(r)}(\theta,t)+\cdots+A_n^{(r)}K_n^{(r)}(\theta,t)}{A_1^{(r)}+A_2^{(r)}+\cdots+A_n^{(r)}},$$

where

$$A_n^{(r)}=\frac{(r+1)(r+2)\cdots(r+n-1)}{(n-1)!},$$

it follows that if

$$K_n^{(r)}(\theta,t)\geqq 0 \qquad (n=1,2,\cdots\cdots),$$

then

$$K_n^{(r+1)}(\theta,t)\geqq 0 \qquad (n=1,2,\cdots\cdots).$$

Consequently we have the theorem:

Theorem II. *When all the curves* $C_n^{(r)}$ $(n=1,2,\cdots\cdots)$ *for any curve* $C$ *lie in the domain* $D$, *all the curves* $C_n^{(r+1)}$ $(n=1,2,\cdots\cdots)$ *have the same nature.*

**6.** As the second application we come to the problem of summability.

When the function $f(t)$ is limited and integrable in the interval

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(1) From $\varphi_0(t)=\pm 1$, it follows that

$$K_1^{(r)}(\theta,t)=1>0.$$

$(0 \leqq t \leqq 1)$ and moreover there exist the $n$ constants $c_0, c_1, \cdots\cdots c_{n-1}$, such that

$$|f(t) - c_0 \varphi_0(t) - c_1 \varphi_1(t) - \cdots\cdots - c_{n-1} \varphi_{n-1}(t)| < \delta, \quad (0 \leqq t \leqq 1),$$

corresponding to any given positive number $\delta$, $f(t)$ is said to lie within *the domain of the orthogonal functions* $\varphi_n(t)$.

Now applying the method of Prof. Haar([1]) to the series (1), we can show that when there exists a fixed positive number $M$, independent of $n$ and $\theta$ $(0 \leqq \theta \leqq 1)$, such that

$$\int_0^1 |K_n^{(r)}(\theta, t)| \, dt < M,$$

the $r^{\text{th}}$ mean of the series (1) corresponding to any function $f(t)$, lying within the domain of the orthogonal functions $\varphi_n(t)$, converges to $f(\theta)$ at every point of the interval $(0 \leqq \theta \leqq 1)$ at which it is continuous.

But if all the curves $C_n^{(r)}$ $(n=1, 2, \cdots\cdots)$ for any curve $C$ lie in the domain $D$, then

$$K_n^{(r)}(\theta, t) \geqq 0, \qquad (n=1, 2, \cdots\cdots) \tag{6}$$

at every point of the domain $(0 \leqq \theta \leqq 1,\ 0 \leqq t \leqq 1)$ at which they are continuous; and

$$\int_0^1 K_n^{(r)}(\theta, t) \, dt = 1, \qquad (n=1, 2, \cdots\cdots); \tag{8}$$

so that

$$\int_0^1 |K_n^{(r)}(\theta, t)| \, dt = 1.$$

Hence we can infer the theorem:

Theorem III. *If all the curves* $C_n^{(r)}$ $(n=1, 2, \cdots\cdots)$ *for any curve* $C$:

$$x = x(\theta), \qquad y = y(\theta), \qquad (0 \leqq \theta \leqq 1),$$

*where* $x(\theta)$ *and* $y(\theta)$ *are functions lying within the domain of the orthogonal functions* $\varphi_n(\theta)$, *lie in the domain* $D$, *then the sequence of curves* $C_n^{(r)}$ $(n=1, 2, \cdots\cdots)$ *converges to that curve* $C$ *at every point of the curve at which it is continuous.*

**7.** Lastly we will give some simple examples.

I. If we take the system of orthogonal functions such as

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([1]) Haar, "Zur Theorie der orthogonalen Funktionensysteme," 1. Mitteilung, Math. Ann., 69 (1910), p. 331.

(i) $$\varphi_n(t)=\sqrt{2}\sin n\pi t, \qquad (n=0, 1, 2, \cdots\cdots);$$

or

(ii) $$\varphi_n(t)=J(\sqrt{\lambda_n}\, t)\left[\int_0^1 J^2(\sqrt{\lambda_n}\, t)\, dt\right]^{-\frac{1}{2}}, \quad (n=0, 1, 2, \cdots\cdots),$$

$J$ being the Bessel function of the $0^{\text{th}}$ order and $\lambda_n$ the $n^{\text{th}}$ roots of the equation

$$J(\sqrt{\lambda})=0,$$

then the identity

$$\varphi_0(t)=1 \quad \text{or} \quad -1$$

does not hold.

In such cases we can choose the curve $C$ for which the domain $D$ can not contain all the curves $C_n^{(r)}$ $(n=1, 2, \cdots\cdots)$ *however great* $r$ *is taken* (See § 3).

II. As the second example, we consider the system of Haar's functions ([1]):

$$\begin{aligned}
\varphi_0(t)&=1 && 0\leqq t\leqq 1;\\
\varphi_n^{(k)}(t)&=0 && 0\leqq t<(2k-2)\,2^{-n},\\
&=\frac{1}{2}\sqrt{2^{n-1}}, && t=(2k-2)\,2^{-n},\\
&=\sqrt{2^{n-1}}, && (2k-2)\,2^{-n}<t<(2k-1)\,2^{-n},\\
&=0, && t=(2k-1)\,2^{-n},\\
&=-\sqrt{2^{n-1}}, && (2k-1)\,2^{-n}<t<2k\cdot 2^{-n},\\
&=-\frac{1}{2}\sqrt{2^{n-1}}, && t=2k\cdot 2^{-n},\\
&=0, && 2k\cdot 2^{-n}<t\leqq 1,\\
& && (k=1, 2, \cdots\cdots, 2^{n-1}).
\end{aligned}$$

Then

$$\varphi_0(t)=1,\ \varphi_1(t),\ \varphi_2^{(1)}(t),\ \varphi_2^{(2)}(t),\ \varphi_3^{(1)}(t),\ \varphi_3^{(2)}(t), \cdots\cdots$$

form a system of orthogonal functions, and it is seen, as Prof. Haar proved, that

$$K_n^{(0)}(\theta, t)\geqq 0, \qquad (n=2, 3, \cdots\cdots),$$
$$0\leqq\theta\leqq 1, \quad 0\leqq t\leqq 1.$$

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([1]) Haar, loc. cit.

Hence we have the theorem:

Theorem IV. *All the sum curves* $C_n^{(0)}$ $(n=1, 2, \cdots\cdots)$ *for any curve* $C$, *corresponding to the series of Haar's function, lie in the domain* $D$.

III. As the third example, we take the system:

$$\varphi_0(t)=1, \quad \varphi_{2n-1}(t)=\sqrt{2}\cos 2n\pi t, \quad \varphi_{2n}(t)=\sqrt{2}\sin 2n\pi t.$$

In this case from the well known inequalities

$$K_n^{(1)}(\theta, t) \geqq 0, \qquad (n=1, 2, \cdots\cdots),$$

we arrive at the theorem due to Prof. Fejér (§ 1).

Lastly we add a remark: For the series of orthogonal functions $\varphi_n(t)$ in general, *the inequalities*

$$K_n^{(1)}(\theta, t) \geqq 0, \qquad (n=2, 3, \cdots),$$

*where*

$$\varphi_0(t)=+1 \quad \text{or} \quad -1$$

*hold good if there exist the inequalities of the doubly symmetric determinants* (1):

$$\begin{vmatrix} 2 & \varphi_1(\theta)\varphi_1(t) & \varphi_2(\theta)\varphi_2(t) & \cdots & \varphi_{n-2}(\theta)\varphi_{n-2}(t) & \varphi_{n-1}(\theta)\varphi_{n-1}(t) \\ \varphi_1(\theta)\varphi_1(t) & 2 & \varphi_1(\theta)\varphi_1(t) & \cdots & \varphi_{n-3}(\theta)\varphi_{n-3}(t) & \varphi_{n-2}(\theta)\varphi_{n-2}(t) \\ \varphi_2(\theta)\varphi_2(t) & \varphi_1(\theta)\varphi_1(t) & 2 & \cdots & \varphi_{n-4}(\theta)\varphi_{n-4}(t) & \varphi_{n-3}(\theta)\varphi_{n-3}(t) \\ \cdots\cdots & \cdots\cdots & \cdots\cdots & \cdots & \cdots\cdots & \cdots\cdots \\ \cdots\cdots & \cdots\cdots & \cdots\cdots & \cdots & \cdots\cdots & \cdots\cdots \\ \varphi_{n-2}(\theta)\varphi_{n-2}(t) & \varphi_{n-3}(\theta)\varphi_{n-3}(t) & \varphi_{n-4}(\theta)\varphi_{n-4}(t) & \cdots & 2 & \varphi_1(\theta)\varphi_1(t) \\ \varphi_{n-1}(\theta)\varphi_{n-1}(t) & \varphi_{n-2}(\theta)\varphi_{n-2}(t) & \varphi_{n-3}(\theta)\varphi_{n-3}(t) & \cdots & \varphi_1(\theta)\varphi_1(t) & 2 \end{vmatrix} \geqq 0,$$

$$(n=2, 3, \cdots\cdots).$$

This is an immediate consequence of the two theorems of Prof. Fejér (2) and Prof. Toeplitz (3).

Ikeda near Ôsaka, May 1918.

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(1) That is, symmetric with respect to the principal and second diagonals. This determinant is centrosymmetric, and also orthosymmetric with respect to the second diagonal.

(2) Fejér, loc. cit., p. 89 (Theorem VI).

(3) Toeplitz, "Über die *Fourier*sche Entwickelung positiver Funktionen," Rend. Palermo, 32 (1911), p. 191; Carathéodory, "Über die Variabilitätsbereich der *Fourier*'schen Konstanten von positiven harmonischen Funktionen," Rend. Palermo, 32 (1911), p. 193.

# A Remark on the Dynamical System with Two Degrees of Freedom,

by

KINNOSUKE OGURA, Ôsaka.

**1.** In his great memoir "Dynamical systems with two degrees of freedom"(1), Prof. G. D. Birkhoff proved that the equations of motion of any dynamical system with two degrees of freedom can be reduced to *the normal form:*

$$(1) \qquad \ddot{x}+\lambda\dot{y}=\frac{\partial\gamma}{\partial x}, \qquad \ddot{y}-\lambda\dot{x}=\frac{\partial\gamma}{\partial y},$$

where $x$, $y$ denote the two coordinates of the dynamical system, $\dot{x}$, $\dot{y}$ their time derivatives, and $\lambda$, $\gamma$ are real analytic functions of $x$ and $y$.

The dynamical system is *reversible* or *irreversible* according as $\lambda$ vanishes identically or not.

The equations (1) admit the first integral

$$(2) \qquad \frac{1}{2}(\dot{x}^2+\dot{y}^2)=\gamma+h,$$

$h$ being an arbitrary constant. In this short note I will confine myself to the case in which $h$ has a definite value.

Now if we interpret $(x, y)$ as the rectangular point coordinates of a plane, and eliminate the four quantities $\dot{x}$, $\dot{y}$, $\ddot{x}$, $\ddot{y}$ among (1), (2) and the identities

$$y'\equiv\frac{dy}{dx}=\frac{\dot{y}}{\dot{x}}, \qquad y''\equiv\frac{d^2y}{dx^2}=\frac{\dot{x}\ddot{y}-\dot{y}\ddot{x}}{\dot{x}^3},$$

we have *the differential equation of the orbits*

$$(3) \qquad y''=(Ay'-B)(1+y'^2)+C(1+y'^2)^{\frac{3}{2}},$$

where we have put

$$A=-\frac{1}{2}\frac{\partial\log(\gamma+h)}{\partial x}, \quad B=-\frac{1}{2}\frac{\partial\log(\gamma+h)}{\partial y}, \quad C=\frac{\lambda}{\sqrt{2(\gamma+h)}}.$$

(1) Trans. Amer. Math. Soc., 18 (1917), p. 201.

This is nothing but the differential equation already treated by me, and the totality of its integral curves has been called *the N-system*(1). Consequently we have the theorem:

*The totality of the orbits of any dynamical (normal) system with two degrees of freedom consists of an N-system; and conversely any N-system is pomposed of all the orbits of a certain normal system with two degrees of freedom.*

**2.** Applying the results previously obtained in this case, we see that the locus of the centres of curvature of all the orbits (3) at any point $P(x, y)$ is the conic having $P$ as a focus, and $\frac{2}{\lambda}\sqrt{2(\gamma+h)}$ as the parameter, and

$$\frac{\partial \log \sqrt{\gamma+h}}{\partial x}(\xi-x)+\frac{\partial \log \sqrt{\gamma+h}}{\partial y}(\eta-y)=1$$

as the directrix corresponding to $P$, where $(\xi, \eta)$ are the current coordinates; and the converse is also true.

But it is known that the directrix is the locus of the centres of curvature of all the orbits, at the point $P$, in the reversible system

$$y''=(Ay'-B)(1-y'^2);$$

and conversely. On the other hand, Prof. Kasner(2) proved that a system of $\infty^2$ curves in a plane, one for each direction at each point of the plane, will constitute the totality of all the orbits for a reversible system (so called the natural family), when and only when, it possesses the two properties that (i) the osculating circles at any given point must form a pencil, and (ii) the two hyperosculating circles contained in such a pencil must be orthogonal.

Therefore we can state the following theorem which gives *the geometrical characterization* for the normal system with two degrees of freedom:

*A system of $\infty^2$ curves in a plane, one for each direction at each point of the plane, will constitute the totality for any normal system with*

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(1) Ogura, "On the integral curves of ordinary differential equations of the second order of a certain type," Tôhoku Math. Journal, 8 (1915), p. 93. See also Mr. K. Kurosu' paper of the same title, Tôhoku Math. Journal, 12 (1917), p. 197; and Ogura, "On a certain system of doubly infinite curves on a surface," Tôhoku Math. Journal, 8 (1915) p. 213.

(2) Kasner, "Natural families of trajectories." Trans. Amer. Math. Soc., 10 (1909) p. 201, Consult with Ogura, "Trajectories in the conservative field of force," Part. I. Tôhoku Math. Journal, 7 (1915), p. 123.

*two degrees of freedom, when and only when, it possesses the two properties such that*

(i) *the centre of curvature at any given point must lie on a conic having that point as a focus; and*

(ii) *the directrix of the conic corresponding to the focus must coincide with the straight line which is the locus of the centres of curvature of all the orbits at the given point for any reversible system (characterized geometrically by* Kasner's *theorem).*

**3.** Moreover we have *the analytical characterization:*

*A necessary and sufficient condition that*

$$y''=\phi(x, y, y')$$

*may be the differential equation to the orbits of the normal system with two degrees of freedom is that the expression*

$$\frac{\partial \phi}{\partial y'}-\frac{3\phi y'}{1+y'^2}$$

*should be the exact differential quotient of a function of $x$ and $y$ with respec to $x$. When the condition is satisfied, we have*

$$\gamma+h=k \exp.\left\{-2\int\left(\frac{\partial \phi}{\partial y'}-\frac{3\phi y'}{1+y'^2}\right)dx\right\},$$

*$k$ being an arbitrary constant.*

Lastly we remark that Prof. Ph. Frank(1) and Prof. Whittaker(2) mentioned *erroneously* the above as the necessar yand *sufficient* condition for *the reversible system.*

Ikeda near Ôsaka, May 1918.

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(1) Frank, "Über die Bahnkurven der Mechanik," Crelle's Journal, 134 (1908), p, 156.

(2) Whittaker, Treatise on the analytical dynamics (2. ed., 1917), p. 407.

# A Contribution to the Question of Linear Dependence in Linear Integral Equations,

by

Louis J. Rouse, Ann Arbor, Mich., U. S. A.

A set of $m$ linear algebraic functions in $n$ variables

$$\text{(a)} \qquad \sum_{j=1}^{n} a_{ij}\, x_j = y_i, \qquad i=1, 2, 3, \cdots\cdots, m$$

is said to be linearly dependent if there exists a set of $m$ constants $c_i$ not all zero such that

$$\sum_{i=1}^{m} c_i\, y_i \equiv 0.$$

If each function $y_i$ of (a) be multiplied through by the corresponding $c_i$ and the sum of the products taken, the above definition is easily seen to be equivalent to

Theorem I. A necessary and sufficient condition that the set (a) be linearly dependent is that there exist a sets of $m$ constants $c_i$ not all zero such that

$$\text{(b)} \qquad \sum_{i=1}^{m} c_i\, a_{ij} = 0, \qquad j=1, 2, 3, \cdots\cdots, n.$$

Again if (b) be regarded as a homogeneous system of $n$ equations in the $c_i$, we know it can have a non-zero solution, $c_1, c_2, \cdots\cdots, c_m$, if and only if the rank of the matrix

$$\| a_{ji} \| \qquad i=1, 2, 3, \cdots\cdots, m; \qquad j=1, 2, 3, \cdots\cdots, n$$

is less than $m$. Hence our definition of linear dependence is also equivalent to

Theorem II. A necessary and sufficient condition that the set (a) be linearly dependent is that the rank of the system of $m$ equations

$$\text{(c)} \qquad \sum_{j=1}^{n} a_{ij}\, x_j = 0, \qquad i=1, 2, 3, \cdots\cdots, m$$

be less than $m$.

Suppose $m<n$. Then from Theorem II it follows that if the func

tions $y_i$, $i=1, 2, 3, \cdots\cdots, m$ be linearly dependent the determinant of the set

$$\text{(d)} \qquad \begin{aligned} &\sum_{j=1}^{n} a_{ij}\, x_j = y_i \qquad && i=1, 2, 3, \cdots\cdots, m, \\ &\sum_{j=1}^{n} a_{hj}\, x_j = y_h \qquad && h=m+1,\ m+2, \cdots\cdots, n, \end{aligned}$$

where the $a_{hj}$ are entirely arbitrary, is zero and there exists a non-zero solution for the system

$$\text{(e)} \qquad \begin{aligned} &\sum_{j=1}^{n} a_{ij}\, x_j = 0 \qquad && i=1, 2, 3, \cdots\cdots, m \\ &\sum_{j=1}^{n} a_{hj}\, x_j = 0 \qquad && h=m+1,\ m+2, \cdots\cdots, n \end{aligned}$$

for the $a_{hj}$ arbitrary. The converse is also true. For suppose there exists a non-zero solution of the system (e) whatever be the last $n-m$ equations. Then we must have

$$A = \begin{vmatrix} a_{11}\, a_{12} \cdots\cdots\cdots\cdots\cdots\cdots a_{1n} \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ a_{m1}\, a_{m2} \cdots\cdots\cdots\cdots\cdots a_{mn} \\ a_{m+1,1}\, a_{m+1,2} \cdots\cdots\cdots\cdots a_{m+1,n} \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ a_{n,1}\, a_{n2} \cdots\cdots\cdots\cdots\cdots\cdots a_{nn} \end{vmatrix} \equiv 0$$

for all $a_{hj}$, $h=m+1,\ m+2, \cdots\cdots, n$; $j=1, 2, 3, \cdots\cdots, n$.

If we denote by $\bar{a}_{i_1 i_2 \cdots i_m}$ the minor of order $m$ formed by taking the elements of the first $m$ rows and the $i_1^{th}, i_2^{th}, \cdots\cdots i_m^{th}$ columns of $A$, and by $A_{i_1 i_2 \cdots i_m}$ the co-factor of $\bar{a}_{i_1 i_2 \cdots i_m}$, the expansion of $A$ according to minors of the first $m$ rows can be written

$$A = \sum_{i_1, i_2, \cdots i_m = 1}^{n} \bar{a}_{i_1 i_2 \cdots i_m} A_{i_1 i_2 \cdots i_m} \equiv 0$$

$$i_1 < i_2 < i_3 < \cdots\cdots < i_m.$$

From this it follows, since the $A_{i_1 i_2 \cdots i_m}$ are not identically zero, that

$$\bar{a}_{i_1 i_2 \cdots i_m} = 0 \qquad \left.\begin{matrix} i_1 \\ i_2 \\ \vdots \\ i_m \end{matrix}\right\} = 1, 2, 3, \cdots\cdots, n.$$

i.e. the rank of the first $m$ equations of (e) must be less than $m$, and we thus have

Theorem III. A necessary and sufficient condition that the set (a) for $m<n$ shall be linearly dependent is that for any arbitrary set of $a_{hj}$, the system of equations (e) have a non-zero solution $x_1, x_2, \cdots\cdots, x_n$, i.e. the determinant of the set (d) be identically zero in the $a_{hj}$.

We propose in this paper to consider linear dependence relative to linear integral expressions involving two or more functions $\phi_1(x)$, $\phi_2(x)$, $\cdots\cdots\phi_n(x)$, i.e. expressions of the type

$$\phi_i(x)-\sum_{j=1}^{n}\int_a^b k_{ij}(x,y)\varphi_j(y)dy=f_i(x),\quad i=1,2,3,\cdots\cdots,m.$$

We shall give as our definition of linear dependence for a set of such linear integral expressions the analogue to Theorem III above, viz:

Definition. A set of $m$ linear integral expressions

$$(1)\qquad \varphi_i(x)-\sum_{j=1}^{n}\int k_{ij}(x,y)\varphi_j(y)dy=f_i(x)\quad i=1,2,3,\cdots\cdots,m$$

is said to be linearly dependent if $n$ functions $\varphi_1(x), \varphi_2(x), \cdots\cdots, \varphi_n(x)$, not all zero, exist which satisfy the system of equations

$$(2)\qquad \begin{aligned} &\varphi_i(x)-\sum_{j=1}^{n}\int k_{ij}(x,y)\,\varphi(y)dy=0, && i=1,2,\cdots\cdots,m.\\ &\varphi_h(x)-\sum_{j=1}^{n}\int k_{hj}(x,y)\,\varphi_j(y)dy=0, && h=m+1,m+2,\cdots\cdots,n,\end{aligned}$$

for $k_{hj}(x,y)$ entirely arbitrary, i.e. if the Fredholm determinant of (2) is identically zero for all $k_{hj}(x,y)$.

In Part I we study the consequences of this definition with respect to a single linear integral expression in two functions

$$(3)\qquad \varphi_1(x)-\int k_{11}(x,y)\,\varphi_1(y)dy-\int k_{12}(x,y)\,\varphi_2(y)dy=f_1(x),$$

in which, for convenience, $\varphi_1(x)$, $\varphi_2(x)$ and $f_1(x)$ are assumed to be continuous on the interval $a\leqq x\leqq b$ and $k_{11}(x,y)$, $k_{12}(x,y)$, continuous in the square $a\leqq x\leqq b$, $a\leqq y\leqq b$(1).

(1) Throughout this paper we shall limit ourselves to functions $\varphi(x)$ and $f(x)$ continuous in $a\leqq x\leqq b$ and functions $k(x,y)$ continuous in the square $a\leqq x\leqq b$, $a\leqq y\leqq b$. For other possible conditions cf. Bôcher, An Introduction to the Study of Integral Equation, p. 3. We shall also assume that our integrations are extended over the interval $a\leqq x\leqq b$.

We develop first in § 1, some interesting expansions of Fredholm determinants, of certain combinations of kernels from which we deduce, in § 2, some necessary and sufficient conditions for linear dependence relative to equation (3). In § 3 we establish for (3) a theorem analogous to Theorem I above.

In Part II we consider a set of $m$ linear integral expressions in $n$ functions

$$(4) \qquad \varphi_i(x) - \sum_{j=1}^{n} \int k_{ij}(x,y)\, \varphi_j(y) dy = f_i(x), \quad i=1, 2, 3, \cdots\cdots, m,$$

to the form of which the more general set

$$(4') \qquad \sum_{j=1}^{n} a_{ij}(x)\, \varphi_j(x) - \sum_{j=1}^{n} \int k_{ij}(x,y)\, \varphi_j(y) dy = f_i(x)$$

$$i=1, 2, 3, \cdots\cdots, m$$

can always be reduced provided there exists an $m$-rowed determinant

$$A(x) = | a_{il}(x) | \qquad i, l, = 1, 2, 3, \cdots\cdots, m$$

which does not vanish on the interval $a \leqq x \leqq b$ (1).

Since the determinantal formulae developed in Part I become very complicated in the general case, we give only a treatment of this case analogous to that of Part I, § 3.

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(1) If we suppose for convenience that the determinant $| a_{il}(x) |$, $(i, l = 1, 2, \cdots m)$ does not vanish on the interval $a \leqq x \leqq b$, then by solving the equations (4′) for $\varphi_1(x), \cdots, \varphi_m(x)$, we get a system of the form

$$\varphi_i(x) + \sum_{h=m+1}^{n} b_{ih}(x)\, \varphi_h(x) - \sum_{l=1}^{m} \int \bar{k}_{il}(x,y)\, \varphi_l(y) dy - \sum_{h=m+1}^{m} \int \bar{k}_{ih}(x,y)\, \varphi_h(y) dy = \bar{f}_i(x)$$

$$i=1, 2, 3, \cdots, m.$$

If now we set

$$\varphi_i(x) + \sum_{h=m+1}^{n} b_{ih}(x)\, \varphi_h(x) = \Phi_i(x) \qquad i=1, 2, 3, \cdots, m$$

$$\varphi_h(x) = \Phi_h(x) \qquad h=m+1, m+2, \cdots, n.$$

Then these can be reduced to

$$\Phi_i(x) - \sum_{j=1}^{n} \int \bar{\bar{k}}_{ij}(x,y)\, \Phi_j(y) dy = \bar{f}_i(x) \qquad i=1, 2, 3, \cdots m,$$

where the $\bar{\bar{k}}_{ij}(x,y)$ are easily expressible in terms of the $\bar{k}_i(x,y)$ and so in terms of the $k_{ij}(x,y)$.

## PART I.

§ 1. Fredholm[1] has suggested a method for finding the Fredholm determinant for a system of $n$ integral equations

$$\varphi_g(x)-\sum_{j=1}^{n}\int k_{gj}(x,y)\,\varphi_j(y)dy=f_g(x),\quad g=1,2,3,\cdots\cdots,n$$

which is equivalent to the formula

$$(1)\qquad F_0[k_{gj}]=1+\sum_{m=1}^{\infty}\frac{(-1)^m}{m!}\sum_{g_1=1}^{n}\int\sum_{g_2=1}^{n}\int\cdots\cdots$$

$$\cdots\cdots\cdots\sum_{g_m=1}^{n}\int K^{g_1g_2\cdots g_m}_{g_1g_2\cdots g_m}\begin{pmatrix}x_1x_2\cdots x_m\\x_1x_2\cdots x_m\end{pmatrix}dx_1\,dx_2\cdots\cdots dx_m$$

where

$$K^{g_1g_2\cdots g_m}_{j_1j_2\cdots j_m}\begin{pmatrix}x_1x_2\cdots x_m\\y_1y_2\cdots y_m\end{pmatrix}=\begin{vmatrix}k_{g_1j_1}(x_1y_1) & k_{g_1j_2}(x_1y_2)\cdots\cdots\cdots\cdots & k_{g_1j_m}(x_1y_m)\\ k_{g_2j_1}(x_2y_1) & k_{g_2j_2}(x_2y_2)\cdots\cdots\cdots\cdots & k_{g_2j_m}(x_2y_m)\\ \cdots & \cdots\cdots\cdots\cdots & \cdots\\ \cdots & \cdots\cdots\cdots\cdots & \cdots\\ k_{g_mj_1}(x_my_1) & k_{g_mj_2}(x_my_2)\cdots\cdots\cdots & k_{g_mj_m}(x_my_m)\end{vmatrix}$$

In the case of the system of two equations

$$(2)\qquad\begin{aligned}&\varphi_1(x)-\int k_{11}(x,y)\,\varphi_1(y)dy-\int k_{12}(x,y)\,\varphi_2(y)dy=f_1(x),\\&\varphi_2(x)-\int k_{21}(x,y)\,\varphi_1(y)dy-\int k_{22}(x,y)\,\varphi_1(y)dy=f_2(x)\end{aligned}$$

(1) becomes

$$(3)\qquad F_0\begin{bmatrix}k_{11}k_{12}\\k_{21}k_{22}\end{bmatrix}=1+\sum_{m=1}^{\infty}\frac{(-1)^m}{m!}\sum_{g_1=1}^{2}\int\sum_{g_2=1}^{2}\int\cdots$$

$$\cdots\sum_{g_m=1}^{2}\int K^{g_1g_2\cdots g_m}_{g_1g_2\cdots g_m}\begin{pmatrix}x_1x_2\cdots x_m\\x_1x_2\cdots x_m\end{pmatrix}dx_1\,dx_2\cdots dx_m=1-\int[(k_{11}(x_1x_1)$$

$$+K_{22}(x_1x_1)]\,dx_1+\frac{1}{2!}\iint\left[K_{11}\begin{pmatrix}x_1x_2\\x_1x_2\end{pmatrix}+2\begin{vmatrix}k_{11}(x_1x_1) & k_{12}(x_1x_2)\\k_{21}(x_2x_1) & k_{22}(x_2x_2)\end{vmatrix}\right.$$

$$\left.+k_{22}\begin{pmatrix}x_1x_2\\x_1x_2\end{pmatrix}\right]dx_1\,dx_2-\frac{1}{3!}\iiint\left[K_{11}\begin{pmatrix}x_1x_2x_3\\x_1x_2x_3\end{pmatrix}\right.$$

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(1) Acta Math. Vol. 27 (1903), p. 378. See also Platrier, Journal de Mathématiques, series 6, Vol. 9 (1913), p. 261.

$$+3\begin{vmatrix} k_{11}(x_1x_1) & k_{11}(x_1x_2) & k_{12}(x_1x_3) \\ k_{11}(x_2x_1) & k_{11}(x_2x_2) & k_{12}(x_2x_3) \\ k_{21}(x_3x_1) & k_{21}(x_3x_2) & k_{22}(x_3x_3) \end{vmatrix} + 3\begin{vmatrix} k_{11}(x_1x_1) & k_{12}(x_1x_2) & k_{12}(x_1x_3) \\ k_{21}(x_2x_1) & k_{22}(x_2x_2) & k_{22}(x_2x_3) \\ k_{21}(x_3x_1) & k_{22}(x_3x_2) & k_{22}(x_3x_3) \end{vmatrix}$$

$$+K_{22}\begin{pmatrix} x_1x_2x_3 \\ x_1x_2x_3 \end{pmatrix}\Bigg]\, dx_1\, dx_2\, dx_3 + \frac{1}{4!}\int\int\int\int\Bigg[K_{11}\begin{pmatrix} x_1x_2x_3x_4 \\ x_1x_2x_3x_4 \end{pmatrix}$$

$$+4\begin{vmatrix} k_{11}(x_1x_1) & k_{11}(x_1x_2) & k_{11}(x_1x_3) & k_{12}(x_1x_4) \\ k_{11}(x_2x_1) & k_{11}(x_2x_2) & k_{11}(x_2x_3) & k_{12}(x_2x_4) \\ k_{11}(x_3x_1) & k_{11}(x_3x_2) & k_{11}(x_3x_3) & k_{12}(x_3x_4) \\ k_{21}(x_4x_1) & k_{21}(x_4x_2) & k_{21}(x_4x_3) & k_{22}(x_4x_4) \end{vmatrix}$$

$$+6\begin{vmatrix} k_{11}(x_1x_1) & k_{11}(x_1x_2) & k_{12}(x_1x_3) & k_{12}(x_1x_4) \\ k_{11}(x_2x_1) & k_{11}(x_2x_2) & k_{12}(x_2x_3) & k_{12}(x_2x_4) \\ k_{21}(x_3x_1) & k_{21}(x_3x_2) & k_{22}(x_3x_3) & k_{22}(x_3x_4) \\ k_{21}(x_4x_1) & k_{21}(x_4x_2) & k_{22}(x_4x_3) & k_{22}(x_4x_4) \end{vmatrix}$$

$$+4\begin{vmatrix} k_{11}(x_1x_1) & k_{12}(x_1x_2) & k_{12}(x_1x_3) & k_{12}(x_1x_4) \\ k_{21}(x_2x_1) & k_{22}(x_2x_2) & k_{22}(x_2x_3) & k_{22}(x_2x_4) \\ k_{21}(x_3x_1) & k_{22}(x_3x_2) & k_{22}(x_3x_3) & k_{22}(x_3x_4) \\ k_{21}(x_4x_1) & k_{22}(x_4x_2) & k_{22}(x_4x_3) & k_{22}(x_4x_4) \end{vmatrix}$$

$$+K_{22}\begin{pmatrix} x_1x_2x_3x_4 \\ x_1x_2x_3x_4 \end{pmatrix} dx_1\, dx_2\, dx_3\, dx_4 - \cdots\cdots\cdots\cdots.$$

By expanding still further according to the rows containing $k_{11}$ and $k_{12}$ and properly collecting terms, it is easily seen that after placing

$$x_{m+1}, x_{m+2}, \cdots\cdots, x_{2m} = y_1, y_2, \cdots\cdots, y_m \text{ and } x_{2m+1}, x_{2m+2}, \cdots\cdots = s_1, s_2, \cdots\cdots$$

(3) becomes

$$(4) \qquad F_\circ\begin{bmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{bmatrix} = F_\circ(k_{11})\; F_\circ(k_{22}) + \sum_{m=1}^{\infty}{}' \frac{(-1)^m}{m\,!}$$

$$\times\int\cdots\int F_m\begin{pmatrix} x_1\cdots\cdots x_m \\ y_1\cdots\cdots y_m \end{pmatrix}; \begin{matrix} k_{12} \\ k_{11} \end{matrix}\Bigg) F_m\begin{pmatrix} y_1\cdots\cdots y_m \\ x_1\cdots\cdots x_m \end{pmatrix}; \begin{matrix} k_{21} \\ k_{22} \end{matrix}\Bigg) dx_1\cdots\cdots dx_m\, dy_1\cdots\cdots dy_m,$$

where

$$F_m\begin{pmatrix} x_1\cdots\cdots x_m \\ y_1\cdots\cdots y_m \end{pmatrix}; \begin{matrix} k_{12} \\ k_{11} \end{matrix}\Bigg) = K_{12}\begin{pmatrix} x_1\cdots\cdots x_m \\ y_1\cdots\cdots y_m \end{pmatrix} + \sum_{n=1}^{\infty}{}' \frac{(-1)^n}{n\,!}$$

$$\times\int\cdots\int\begin{vmatrix} k_{12}(x_1\,y_1)\cdots & k_{12}(x_1\,y_m) & k_{11}(x_1\,s_1)\cdots & k_{11}(x_1\,s_n) \\ \cdots\cdots\cdots & \cdots\cdots\cdots & \cdots\cdots & \cdots\cdots \\ \cdots\cdots\cdots & \cdots\cdots\cdots & \cdots\cdots & \cdots\cdots \\ k_{12}(x_m\,y_1)\cdots & k_{12}(x_m\,y_m) & k_{11}(x_m\,s_1)\cdots & k_{11}(x_m\,s_n) \\ k_{12}(s_1\,y_1)\cdots & k_{12}(s_1\,y_m) & k_{11}(s_1\,s_1)\cdots & k_{11}(s_1\,s_n) \\ \cdots\cdots\cdots & \cdots\cdots\cdots & \cdots\cdots & \cdots\cdots \\ \cdots\cdots\cdots & \cdots\cdots\cdots & \cdots\cdots & \cdots\cdots \\ k_{12}(s_n\,y_1)\cdots & k_{12}(s_n\,y_m) & k_{11}(s_n\,s_1)\cdots & k_{11}(s_n\,s_n) \end{vmatrix} ds_1\,ds_2\cdots ds_n$$

and $F_m\left(\begin{matrix} y_1\cdots\cdots y_m \\ x_1\cdots\cdots x_m \end{matrix};\begin{matrix} k_{21} \\ k_{22}\end{matrix}\right)$ is similarly expressed.

If we expand this last expression according to minors of the first $m$ columns and again collect terms properly, we obtain another expression for $F_m\left(\begin{matrix} x_1\cdots\cdots x_m \\ y_1\cdots\cdots y_m \end{matrix};\begin{matrix} k_{12} \\ k_{11}\end{matrix}\right)$, viz:

$$F_m\left(\begin{matrix} x_1\cdots\cdots x_m \\ y_1\cdots\cdots y_m \end{matrix};\begin{matrix} k_{12} \\ k_{11}\end{matrix}\right)=F_\circ(k_{11})\,K_{12}\left(\begin{matrix} x_1\cdots\cdots x_m \\ y_1\cdots\cdots y_m \end{matrix}\right)$$

$$+\int\sum_{i=1}^{m}F_1\left(\begin{matrix} x_i \\ s_1 \end{matrix};k_{11}\right)K_{12}\left(\begin{matrix} x_1\cdots x_{i-1}\,s_1\,x_{i+1}\cdots x_m \\ y_1\cdots\cdots\;\;y_i\;\;\cdots\cdots y_m \end{matrix}\right)ds_1+\cdots\cdots$$

$$+\frac{1}{(q!)^2}\int\cdots\int\sum_{i_1=1}^{m}\cdots\cdots\sum_{i_q=1}^{m}F_q\left(\begin{matrix} xi_1\cdots\cdots xi_q \\ s_1\;\cdots\cdots s_q \end{matrix};k_{11}\right)$$

$$\times K_{12}\left(\begin{matrix} x_1\cdots xi_{1-1}\,s_1\,xi_{1+1}\cdots xi_{g-1}\,s_g\,xi_{q+1}\cdots\;x_m \\ y_1\;\cdots\cdots\cdots yi_1\;\cdots\cdots\cdots yi_q\;\cdots\cdots\cdots y_m \end{matrix}\right)ds_1\cdots\cdots ds_q+\cdots\cdots$$

$$+\frac{1}{m\,!}\int\cdots\int F_m\left(\begin{matrix} x_1\cdots\cdots x_m \\ s_1\cdots\cdots s_m \end{matrix};k_{11}\right)K_{12}\left(\begin{matrix} s_1\;\cdots\;s_m \\ y_1\cdots\cdots y_m \end{matrix}\right)ds_1\cdots\cdots ds_m.$$

In a similar way we obtain

$$F_m\left(\begin{matrix} y_1\cdots\cdots y_m \\ x_1\cdots\cdots x_m \end{matrix};\begin{matrix} k_{21} \\ k_{22}\end{matrix}\right)=F_\circ(k_{22})\,K_{21}\left(\begin{matrix} y_1\cdots\cdots y_m \\ x_1\cdots\cdots x_m \end{matrix}\right)$$

$$+\int\sum_{i=1}^{m}F_1\left(\begin{matrix} y_i \\ s_1 \end{matrix};k_{22}\right)K_{21}\left(\begin{matrix} y_1\cdots y_{i-1}s_1\,y_{i+1}\cdots y_m \\ x_1\cdots\cdots\;x_i\;\;\cdots\;x_m \end{matrix}\right)ds_1+\cdots\cdots$$

$$+\frac{1}{(q.!)^2}\int\cdots\int\sum_{i_1=1}^{m}\cdots\cdots\sum_{i_q=1}^{m}F_q\left(\begin{matrix} yi_1\cdots\cdots yi_q \\ s_1\;\cdots\cdots s_q \end{matrix};k_{22}\right)$$

$$\times K_{21}\left(\begin{matrix} y_1\cdots yi_1-1\;s_1\;\;yi_1+1\cdots yi_q-1\;s_q\,yi_q+1\cdots y_m \\ x_1\;\cdots\cdots\cdots xi_1\cdots\cdots\cdots\cdots xi_q\cdots\cdots\;\;\cdots x_m \end{matrix}\right)ds_1\cdots\cdots ds_q+\cdots\cdots$$

$$+\frac{1}{m!}\int\cdots\int F_m\begin{pmatrix} y_1\cdots\cdots y_m \\ s_1\cdots\cdots s_m \end{pmatrix}; k_{22}\Big) K_{21}\begin{pmatrix} s_1\cdots\cdots s_m \\ y_1\cdots\cdots y_m \end{pmatrix} ds_1\cdots\cdots ds_m.$$

If now we substitute these values for

$F_m\begin{pmatrix} x_1\cdots\cdots x_m \\ y_1\cdots\cdots y_m \end{pmatrix}; \begin{matrix} k_{12} \\ k_{11} \end{matrix}\Big)$ and $F_m\begin{pmatrix} y_1\cdots\cdots y_m \\ x_1\cdots\cdots x_m \end{pmatrix}; \begin{matrix} k_{21} \\ k_{22} \end{matrix}\Big)$ in (4) we get another formula for $F_\circ\begin{bmatrix} k_{11}\, k_{12} \\ k_{21}\, k_{22} \end{bmatrix}$, viz:

$$(5)\qquad F_\circ\begin{bmatrix} k_{11}\, k_{12} \\ k_{21}\, k_{22} \end{bmatrix} = F_\circ(k_{11})\; F_\circ(k_{22}) - \int\int\Big[F_\circ(k_{11})\, k_{12}(x_1\, y_1)$$

$$+\int F_1\begin{pmatrix} x_1 \\ s_1 \end{pmatrix}; k_{11}\Big) K_{12}\begin{pmatrix} s_1 \\ y_1 \end{pmatrix} ds_1\Big]\Big[F_\circ(k_{22})\, K_{21}\begin{pmatrix} y_1 \\ x_1 \end{pmatrix}$$

$$+\int F_1\begin{pmatrix} y_1 \\ s_1 \end{pmatrix}; k_{22}\Big) K_{21}\begin{pmatrix} s_1 \\ x_1 \end{pmatrix} ds_1\Big] dx_1\, dy_1$$

$$+\frac{1}{(2!)^2}\int\int\int\int\Big[F_\circ(k_{11})\; K_{12}\begin{pmatrix} x_1\, x_2 \\ y_1\, y_2 \end{pmatrix}$$

$$+\int \sum_{i_1=1}^{2} F_1\begin{pmatrix} x_i \\ s_1 \end{pmatrix}; k_{11}\Big) K_{12}\begin{pmatrix} x_1\, s_1 \\ y_1\, y_2 \end{pmatrix} ds_1$$

$$+\frac{1}{2!}\int\int F_2\begin{pmatrix} x_1\, x_2 \\ s_1\, s_2 \end{pmatrix}; k_{11}\Big) K_{12}\begin{pmatrix} s_1\, s_2 \\ y_1\, y_2 \end{pmatrix} ds_1\, ds_2\Big]\Big[F_\circ(k_{22})\; K_{21}\begin{pmatrix} y_1\, y_2 \\ x_1\, x_2 \end{pmatrix}$$

$$+\int \sum_{i=1}^{2} F_1\begin{pmatrix} y_i \\ s_1 \end{pmatrix}; k_{22}\Big) K_{21}\begin{pmatrix} y_1\, s_1 \\ x_1\, x_2 \end{pmatrix} ds_1$$

$$+\frac{1}{2!}\int\int F_2\begin{pmatrix} y_1\, y_2 \\ s_1\, s_2 \end{pmatrix}; k_{22}\Big) K_{21}\begin{pmatrix} s_1\, s_2 \\ x_1\, x_2 \end{pmatrix} ds_1\, ds_2\Big]\; dx_1\, dx_2\, dy_1\, dy_2 + \cdots\cdots$$

$$+\frac{(-1)^m}{(m!)^2}\int\cdots\int\Big[F_\circ(k_{11})\; K_{12}\begin{pmatrix} x_1\cdots\cdots x_m \\ y_1\cdots\cdots y_m \end{pmatrix}$$

$$+\int \sum_{i=1}^{m} F_1\begin{pmatrix} x_i \\ s_1 \end{pmatrix}; k_{11}\Big) K_{12}\begin{pmatrix} x_1\cdots x_{i-1}\, s_1\, x_{i+1}\cdots x_m \\ y_1\cdots\cdots\cdots y_i\cdots\cdots y_m \end{pmatrix} ds_1 + \cdots\cdots$$

$$+\frac{1}{(q!)^2}\int\cdots\int \sum_{i_1=1}^{m}\cdots\cdots\sum_{i_q=1}^{m} F_q\begin{pmatrix} x_{i_1}\cdots\cdots x_{i_q} \\ s_1\cdots\cdots s_q \end{pmatrix}; k_{11}\Big)$$

$$\times K_{12}\begin{pmatrix} x_1\cdots x_{i_1-1}\; s_1\, x_{i_1+1}\cdots x_{i_q-1}\; s_q\, x_{i_q+1}\cdots x_m \\ y_1\cdots\cdots\cdots y_{i_1}\cdots\cdots\cdots\cdots y_{i_q}\cdots\cdots\cdots y_m \end{pmatrix} ds_1\cdots\cdots ds_q + \cdots\cdots$$

$$+\frac{1}{m!}\int\cdots\int F_m\begin{pmatrix} x_1\cdots\cdots x_m \\ s_1\cdots\cdots s_m \end{pmatrix}; k_{11}\Big) K_{12}\begin{pmatrix} s_1\cdots\cdots s_m \\ y_1\cdots\cdots y_m \end{pmatrix} ds_1\cdots\cdots ds_m\Big]$$

$$\times\Bigg[F_\circ(k_{22})\,K_{21}\begin{pmatrix} y_1\cdots\cdots y_m \\ x_1\cdots\cdots x_m\end{pmatrix}$$

$$+\int\sum_{i=1}^{m} F_1\begin{pmatrix} y_i \\ s_1\end{pmatrix};k_{22}\Big)\;K_{21}\begin{pmatrix} y_1\cdots y_{i-1}\,s_1\,y_{i+1}\cdots y_m \\ x_1\cdots\cdots\;\; x_i\cdots\cdots\cdots x_m\end{pmatrix}ds_1+\cdots\cdots$$

$$+\frac{1}{(q\,!)^2}\int\cdots\int\sum_{i_1=1}^{m}\cdots\cdots\sum_{i_q=1}^{m} F_q\begin{pmatrix} y_{i_1}\cdots\cdots y_{i_q} \\ s_1\;\cdots\cdots s_q\end{pmatrix};k_{22}\Big)$$

$$\times K_{21}\begin{pmatrix} y_1\cdots y_{i_1-1}\,s_1\,y_{i_1+1}\cdots y_{i_q-1}\,s_q\,y_{i_q+1}\cdots y_m \\ x_1\cdots\cdots\cdots x_{i_1}\;\cdots\cdots\cdots\cdots\;x_{i_q}\cdots\cdots\cdots x_m\end{pmatrix}ds_1\cdots\cdots ds_q+\cdots\cdots$$

$$+\frac{1}{m\,!}\int\cdots\cdots\int F_m\begin{pmatrix} y_1\cdots\cdots y_m \\ s_1\cdots\cdots s_m\end{pmatrix};k_{22}\Big)$$

$$\times K_{21}\begin{pmatrix} s_1\cdots\cdots s_m \\ x_1\cdots\cdots x_m\end{pmatrix}ds_1\cdots\cdots ds_m\Bigg]\,dx_1\cdots\cdots dx_m\,dy_1\cdots\cdots dy_m+\cdots\cdots.$$

By expanding (3) according to the columns containing $k_{11}(x,y)$ and $k_{21}(x,y)$ we obtain in a similar manner the formula

$$(4')\qquad F_\circ\begin{bmatrix} k_{11} & k_{12} \\ k_{21} & k_{22}\end{bmatrix}=F_\circ(k_{11})\;F_\circ(k_{22})+\sum_{m=1}^{\infty}\frac{(-1)^m}{(m\,!)^2}$$

$$\times\int\cdots\int\bar{F}_m\begin{pmatrix} y_1\cdots\cdots y_m \\ x_1\cdots\cdots x_m\end{pmatrix};\begin{matrix} k_{21} \\ k_{11}\end{matrix}\Big)\;\bar{F}_m\begin{pmatrix} x_1\;\cdots\cdots x_m \\ y_1\cdots\cdots y_m\end{pmatrix};\begin{matrix} k_{12} \\ k_{22}\end{matrix}\Big)\;dx_1\cdots\cdots dx_m\,dy_1\cdots\cdots dy_m$$

where

$$\bar{F}_m\begin{pmatrix} x_1\;\cdots\;x_m \\ y_1\cdots\cdots y_m\end{pmatrix};\begin{matrix} k_{12} \\ k_{22}\end{matrix}\Big)=K_{12}\begin{pmatrix} x_1\cdots\cdots x_m \\ y_1\cdots\cdots y_m\end{pmatrix}-\sum_{n=1}^{\infty}\int\cdots\int\frac{(-1)^n}{n\,!}$$

$$\times\begin{vmatrix} k_{12}(x_1\,y_1)\cdots\cdots k_{12}(x_1\,y_m) & k_{12}(x_1\,s_1)\cdots\cdots k_{12}(x_1\,s_n) \\ \cdots\cdots\cdots\cdots\cdots\cdots & \cdots\cdots\cdots\cdots\cdots\cdots \\ \cdots\cdots\cdots\cdots\cdots\cdots & \cdots\cdots\cdots\cdots\cdots\cdots \\ k_{12}(x_m\,y_1)\cdots\cdots k_{12}(x_m\,y_m) & k_{12}(x_m\,s_1)\cdots\cdots k_{12}(x_m\,s_n) \\ k_{22}(s_1\,y_1)\cdots\cdots k_{22}(s_1\,y_m) & k_{22}(s_1\,s_1)\cdots\cdots k_{22}(s_1\,s_n) \\ \cdots\cdots\cdots\cdots\cdots\cdots & \cdots\cdots\cdots\cdots\cdots\cdots \\ \cdots\cdots\cdots\cdots\cdots\cdots & \cdots\cdots\cdots\cdots\cdots\cdots \\ k_{22}(s_n\,y_1)\cdots\cdots k_{22}(s_n\,y_m) & k_{22}(s_n\,s_1)\cdots\cdots k_{22}(s_n\,s_n) \end{vmatrix}ds_1\cdots\cdots ds_n$$

and $\bar{F}_m\begin{pmatrix} y_1\cdots y_m \\ x_1\cdots x_m\end{pmatrix};\begin{matrix} k_{21} \\ k_{11}\end{matrix}\Big)$ is similarly expressed.

If in (4′) we place $k_{21}=k_{11}$ and $k_{22}=k_{12}$ we obtain an interesting formula for the Fredholm determinant of the sum of two kernels([1]), viz:

$$(6)\quad F_0(k_{11}+k_{12})=F_0(k_{11})\,F_0(k_{12})+\sum_{m=1}^{\infty}\frac{(-1)^m}{(m!)^2}$$

$$\times\int\cdots\cdots\int F_m\left(\begin{matrix}x_1\cdots x_m\\ y_1\cdots y_m\end{matrix};k_{11}\right)F_m\left(\begin{matrix}y_1\cdots y_m\\ x_1\cdots x_m\end{matrix};k_{12}\right)dx_1\cdots dx_m\,dy_1\cdots dy_m\,.$$

This last formula has here been deduced from an expression for $F_0\begin{bmatrix}k_{11} & k_{12}\\ k_{21} & k_{22}\end{bmatrix}$. It can in turn be used to obtain another formula for $F_0\begin{bmatrix}k_{11} & k_{12}\\ k_{21} & k_{22}\end{bmatrix}$. To do this let us replace $k_{12}(y,x)$ by $\lambda\int k_{12}(y,s)\,k_a(s,x)\,ds$ in (6). We then have

$$F_0(k_{11}+\lambda\int k_{12}\,k_a)=F_0(k_{11})\cdot F_0(\lambda\int k_{12}\,k_a)+\sum_{m=1}^{\infty}\frac{(-1)^m}{(m!)^2}$$

$$\times\int\cdots\cdots\int F_m\left(\begin{matrix}x_1\cdots x_m\\ y_1\cdots y_m\end{matrix};k_{11}\right)\cdot F_m\left(\begin{matrix}y_1\cdots y_m\\ x_1\cdots x_m\end{matrix};\lambda\int k_{12}\,k_a\right)dx_1\cdots dx_m\,dy_1\cdots dy_m\,.$$

By expanding this expression still further, making frequent use of the reduction formula([2])

$$\left|\int K'(x_i\,s_i)\,K''(s_i\,y_j)\,ds_i\right|$$

$$=\frac{1}{n!}\int\cdots\int K'\left(\begin{matrix}x_1\cdots x_n\\ s_1\cdots s_n\end{matrix}\right)K''\left(\begin{matrix}s_1\cdots s_n\\ y_1\cdots y_n\end{matrix}\right)ds_1\cdots ds_n\qquad \begin{matrix}i=1,2,3,\cdots n,\\ j=1,2,3,\cdots n,\end{matrix}$$

and collecting terms according to powers of $\lambda$ we obtain a formula for $F_0(k_{11}+\lambda\int k_{12}\,k_a)$, viz:

$$(7)\quad F_0(k_{11}+\lambda\int k_{12}\,k_a)=F_0(k_{11})$$

$$-\lambda\iint\left[F_0(k_{11})\,k_{12}(x_1\,s_1)+\int F_1\left(\begin{matrix}x_1\\ y_1\end{matrix};k_{11}\right)K_{12}\left(\begin{matrix}y_1\\ s_1\end{matrix}\right)dy_1\right]^k K_a(s_1\,x_1)\,dx_1\,ds_1$$

$$+\frac{\lambda^2}{(2!)^2}\iiiint\left[F_0(k_{11})\,K_{12}\left(\begin{matrix}x_1x_2\\ s_1\;s_2\end{matrix}\right)+\int\sum_{i=1}^{2}F_1\left(\begin{matrix}x_i\\ y_1\end{matrix};k_{11}\right)K_{12}\left(\begin{matrix}y_1x_2\\ s_1\;s_2\end{matrix}\right)dy_1\right.$$

$$\left.+\frac{1}{2!}\iint F_2\left(\begin{matrix}x_1\,y_2\\ x_1\,y_2\end{matrix};k_{11}\right)K_{12}\left(\begin{matrix}y_1\,y_2\\ s_1\;s_2\end{matrix}\right)dy_1\,dy_2\right]K_a\left(\begin{matrix}s_1\;s_2\\ x_1\,x_2\end{matrix}\right)dx_1\,dx_2\,ds_1\,ds_2$$

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([1]) Goursat, Ann. de la Faculté des Sciences de Toulouse, Series 2. Vol. 10 (1908), p. 27, has obtained a special case of this formula viz: where $k_1(x,y)$ and $k_2(x,y)$ are orthogonal kernels, in which case it reduces to $F_0(k_1+k_2)=F_0(k_1)\cdot F_0(k_2)$.

([2]) C.f. Schur, Mathematische Annalen, vol. 67 (1909) p. 319; also Landsberg, ibid, vol 69 (1910) p. 221.

$$+\cdots+\frac{(-\lambda)^m}{(m!)^2}\int\cdots\int\Bigg[F_0(k_{11})K_{12}\begin{pmatrix}x_1\cdots x_m\\ s_1\cdots s_m\end{pmatrix}$$

$$+\int\sum_{i=1}^{m}F_1\begin{pmatrix}x_i\\ y_1\end{pmatrix};k_{11}\Big)K_{12}\begin{pmatrix}x_1\cdots x_{i+1}\,y_1x_{i+1}\cdots x_m\\ s_1\cdots\cdots s_i\cdots\cdots s_m\end{pmatrix}dy_1+\cdots\cdots$$

$$+\frac{1}{(q!)^2}\int\cdots\int\sum_{i_1=1}^{m}\cdots\sum_{i_q=1}^{m}F_q\begin{pmatrix}x_{i_1}\cdots x_{i_q}\\ y_1\cdots y_q\end{pmatrix};k_{11}\Big)$$

$$\times K_{12}\begin{pmatrix}x_1\cdots x_{i-1}\,y_1\,x_{i_2+1}\cdots x_{i_q-1}\,y_q\,x_{i_q+1}\cdots\cdots x_m\\ s_1\cdots\cdots s_{i_1}\cdots\cdots\cdots\cdots s_{i_q}\cdots\cdots\cdots s_m\end{pmatrix}dy_1\cdots dy_q+\cdots\cdots$$

$$+\frac{1}{m!}\int\cdots\int F_m\begin{pmatrix}x_1\cdots x_m\\ y_1\cdots y_m\end{pmatrix};k_{11}\Big)K_{12}\begin{pmatrix}y_1\cdots y_m\\ s_1\cdots s_m\end{pmatrix}dy_1\cdots dy_m\Bigg]$$

$$\times K_a\begin{pmatrix}s_1\cdots s_m\\ x_1\cdots x_m\end{pmatrix}dx_1\cdots\cdots dx_m\,ds_1\cdots\cdots ds_m+\cdots\cdots\cdots\cdots.$$

Suppose $F_0(k_{22})\neq 0$ and place

$$k_a(y_1\,x_1)=k_{21}(y_1\,x_1)+\int K_{22}(y_1\,s_1)\,k_{21}(s_1\,x_1)\,ds_1$$

where $K_{22}(y_1\,s_1)$ is the reciprocal of $k_{22}(y_1\,s_1)$, i. e.

$$K_{22}(y_1\,s_1)=\frac{F_1\begin{pmatrix}y_1\\ s_1\end{pmatrix};k_{22}\Big)}{F_0(k_{22})}.$$

If we make use of a second reduction formula([1])

$$\frac{F_q\begin{pmatrix}y_1\cdots y_q\\ s_1\cdots s_q\end{pmatrix};k_{22}\Big)}{F_0(k_{22})}=\begin{vmatrix}\dfrac{F_1\begin{pmatrix}y_1\\ s_1\end{pmatrix};k_{22}\Big)}{F_0(k_{22})} & \cdots\cdots\cdots & \dfrac{F_1\begin{pmatrix}y_1\\ s_q\end{pmatrix};k_{22}\Big)}{F_0(k_{22})}\\ \cdots & \cdots & \cdots\\ \cdots & \cdots & \cdots\\ \dfrac{F_1\begin{pmatrix}y_q\\ s_1\end{pmatrix};k_{22}\Big)}{F_0(k_{22})} & \cdots\cdots\cdots & \dfrac{F_1\begin{pmatrix}y_q\\ s_q\end{pmatrix};k_{22}\Big)}{F_0(k_{22})}\end{vmatrix}$$

and also the formula for $|\int K'\,K''|$ given above, $K_a\begin{pmatrix}y_1\cdots y_q\\ s_1\cdots s_q\end{pmatrix}$ can be reduced to

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([1]) c. f. Plemelj, Monatshefte für Mathematik und Physik, Vol. 15 (1904) p. 101. See also Platrier, loc. cit. p. 249.

$$K_{21}\begin{pmatrix} y_1 \cdots y_m \\ x_1 \cdots x_m \end{pmatrix} + \int \sum_{i=1}^{m} \frac{F_1\begin{pmatrix} y_i \\ s_1 \end{pmatrix}; k_{22}}{F_0(k_{22})} K_{21}\begin{pmatrix} y_1 \cdots y_{i-1}\, s_1\, y_{i+1} \cdots y_m \\ x_1 \cdots\cdots\; x_i \cdots\cdots x_m \end{pmatrix} ds_1 + \cdots\cdots$$

$$+ \frac{1}{(q!)^2} \int \cdots \int \sum_{i_1=1}^{m} \cdots \sum_{i_q=1}^{m} \frac{F_q\begin{pmatrix} y_{i_1} \cdots y_{i_q} \\ s_1 \cdots s_q \end{pmatrix}; k_{22}}{F_0(k_{22})}$$

$$\times K_{21}\begin{pmatrix} y_1 \cdots\cdots y_{i_1-1}\, s_1\, y_{i_1+1} \cdots\cdots y_{i_q-1}\, s_q\, y_{i_q+1} \cdots y_m \\ x_1 \cdots\cdots\cdots x_{i_1} \cdots\cdots\cdots\cdots\cdots x_{i_q} \cdots\cdots\cdots x_m \end{pmatrix} ds_1 \cdots ds_q + \cdots\cdots$$

$$+ \frac{1}{m!} \int \cdots \int \frac{F_m\begin{pmatrix} y_1 \cdots y_m \\ s_1 \cdots s_m \end{pmatrix}; k_{22}}{F_0(k_{22})} K_{21}\begin{pmatrix} s_1 \cdots s_m \\ x_1 \cdots x_m \end{pmatrix} ds_1 \cdots ds_m .$$

Substituting this expression for $K_a\begin{pmatrix} y_1 \cdots y_q \\ s_1 \cdots s_q \end{pmatrix}$ in (7) and comparing the result with formula (5) we see that

$$F_0(k_{11} + \int k_{12}(k_{21} + \int K_{22}\, k_{21})) = \frac{F_0\begin{bmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{bmatrix}}{F_0(k_{22})} .$$

Hence we have the formula

$$(8) \qquad F_0\begin{bmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{bmatrix} = F_0(k_{11} + \int k_{12}(k_{21} + \int K_{22}\, k_{21}))\, F_0(k_{22}) ,$$

where $K_{22}(x, y)$ is the reciprocal of $k_{22}(x, y)$.

Again, if we apply the first reduction formula above to

$$F_0(\int k'\, k'') = 1 + \sum_{m=1}^{\infty} \frac{(-1)^m}{m!} \int \cdots \int | \int k'\, k'' |\, dx_1 \cdots dx_m$$

we obtain

$$(9) \quad F_0(\int k'\, k'') = 1 + \sum_{m=1}^{\infty} \frac{(-1)^m}{(m!)^2}$$

$$\times \int \cdots\cdots \int K'\begin{pmatrix} x_1 \cdots x_m \\ y_1 \cdots y_m \end{pmatrix} K''\begin{pmatrix} y_1 \cdots x_m \\ x_1 \cdots y_m \end{pmatrix} dx_1 \cdots dx_m\, dy_1 \cdots dy_m .$$

Suppose $F_0(k_{11}) \neq 0$ and $F_0(k_{22}) \neq 0$. Then from (9) it follows that for

$$K'(x, y) = k_{12}(x, y) + \int K_{11}(x, s)\, k_{12}(s, y)\, ds$$

and

$$K''(x, y) = k_{21}(x, y) + \int K_{22}(x, s)\, k_{21}(s, y)\, ds$$

where $K_{11}(x, y)$ and $K_{22}(x, y)$ are the reciprocals of $k_{11}(x, y)$ and $k_{22}(x, y)$ respectively i. e.

$$K_{11}(x, y) = \frac{F_1\left(\begin{matrix} x \\ y \end{matrix}; k_{11}\right)}{F_0(k_{11})}$$

and

$$K_{22}(x, y) = \frac{F_1\left(\begin{matrix} x_1 \\ y_1 \end{matrix}; k_{22}\right)}{F_0(k_{22})},$$

we have after applying the second reduction formula above and comparing the result with (5)

$$F_0(\int (k_{12} + \int K_{11}\, k_{12})(k_{21} + \int K_{22}\, k_{21})) = \frac{F_0\begin{bmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{bmatrix}}{F_0(k_{11}) F_0(k_{22})};$$

i.e. we have still another formula for $F_0\begin{bmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{bmatrix}$, viz:

$$(10) \qquad F_0\begin{bmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{bmatrix} = F_0(\int (k_{12} + \int K_{11}\, k_{12})(k_{21} + \int K_{22}\, k_{21})) \cdot F_0(k_{11}) \cdot F_0(k_{22}).$$

We have then the following formulae for the developments of the Fredholm determinants of the various combinations of kernels which we have considered.

I. $$F_0(k_1 + k_2) = F_0(k_1)\, F_0(k_2) + \sum_{m=1}^{\infty} \frac{(-1)^m}{(m!)^2}$$

$$\times \int \cdots\cdots \int F_m\left(\begin{matrix} x_1 \cdots x_m \\ y_1 \cdots y_m \end{matrix}; k_1\right) F_m\left(\begin{matrix} y_1 \cdots y_m \\ x_1 \cdots x_m \end{matrix}; k_2\right) dx_1 \cdots dx_m\, dy_1 \cdots dy_m.$$

II. $$F_0(k_1 + \lambda \int k_2\, k_a) = F_0(k_1) - \lambda \iint \Big[ F_0(k_1)\, k_2(x_1\, s_1)$$

$$+ \int F_1\left(\begin{matrix} x_1 \\ y_1 \end{matrix}; k_1\right) K_2\left(\begin{matrix} y_1 \\ s_1 \end{matrix}\right) dy_1 \Big]\, k_a(s_1\, x_1)\, dx_1\, ds_1$$

$$+ \frac{\lambda^2}{(2!)^2} \iiiint F_0(k_1)\, K_2\left(\begin{matrix} x_1 x_2 \\ s_1\, s_2 \end{matrix}\right) + \int \sum_{i=1}^{2} F_1\left(\begin{matrix} x_i \\ y_1 \end{matrix}; k_1\right) K_2\left(\begin{matrix} y_1 x_2 \\ s_1\, s_1 \end{matrix}\right) dy_1$$

$$+\frac{1}{2!}\int\int F_2\begin{pmatrix} x_1\, x_2 \\ y_1\, y_2 \end{pmatrix}; k_1\Big) K_2\begin{pmatrix} y_1\, y_2 \\ s_1\, s_2 \end{pmatrix} dy_1\, dy_2\Big] K_a\begin{pmatrix} s_1\, s_2 \\ x_1\, x_2 \end{pmatrix} dx_1\, dx_2\, ds_1\, ds_2$$

$$+\cdots\cdots+\frac{(-\lambda)^m}{(m!)}\int\cdots\int\Big[F_0(k_1)\, K_2\begin{pmatrix} x_1\cdots x_m \\ s_1\cdots s_m \end{pmatrix}$$

$$+\int\sum_{i=1}^{m} F_1\begin{pmatrix} x_i \\ y_1 \end{pmatrix}; k_1\Big) K_2\begin{pmatrix} x_1\cdots x_{i-1}\, y_1\, x_{i+1}\cdots x_m \\ s_1\cdots\cdots s_i\cdots\cdots s_m \end{pmatrix} dy_1+\cdots\cdots$$

$$+\frac{1}{(q!)^2}\int\cdots\int\sum_{i_1=1}^{m}\cdots\sum_{i_q=1}^{m} F_q\begin{pmatrix} x_{i_1}\cdots x_{i_q} \\ y_1\cdots y_q \end{pmatrix}; k_1\Big)$$

$$\times K_2\begin{pmatrix} x_1\cdots x_{i_1-1}\, y_1\, x_{i_1+1}\cdots x_{i_q-1}\, y_q\, x_{i_q+1}\cdots x_m \\ s_1\cdots\cdots s_{i_1}\cdots\cdots\cdots\cdots s_{i_q}\cdots\cdots s_m \end{pmatrix} dy_1\cdots dy_q+\cdots\cdots$$

$$+\frac{1}{m!}\int\cdots\int F_m\begin{pmatrix} x_1\cdots x_m \\ y_1\cdots y_m \end{pmatrix}; k_1\Big) K_2\begin{pmatrix} y_1\cdots y_m \\ s_1\cdots s_m \end{pmatrix} dy_1\cdots dy_m\Big]$$

$$\times K_a\begin{pmatrix} s_1\cdots s_m \\ x_1\cdots x_m \end{pmatrix} dx_1\cdots dx_m\, ds_1\cdots ds_m\,.$$

III. $$F_0\begin{bmatrix} k_{11}\, k_{12} \\ k_{21}\, k_{22} \end{bmatrix}=F_0(k_{11})\, F_0(k_{22})+\sum_{m=1}^{\infty}\frac{(-1)^m}{(m!)^2}$$

$$\times\int\cdots\cdots\int F_m\begin{pmatrix} x_1\cdots x_m \\ y_1\cdots y_m \end{pmatrix}; \begin{matrix} k_{12} \\ k_{11} \end{matrix}\Big) F_m\begin{pmatrix} y_1\cdots y_m \\ x_1\cdots x_m \end{pmatrix}; \begin{matrix} k_{21} \\ k_{22} \end{matrix}\Big) dx_1\cdots dx_m\, dy_1\cdots dy_m$$

and similar one,

III′. $$F_0\begin{bmatrix} k_{11}\, k_{11} \\ k_{21}\, k_{22} \end{bmatrix}=F_0(k_{11})\, F_0(k_{22})+\sum_{m=1}^{\infty}\frac{(-1)^m}{(m!)^2}$$

$$\times\int\cdots\cdots\int \overline{F}_m\begin{pmatrix} y_1\cdots y_m \\ x_1\cdots x_m \end{pmatrix}; \begin{matrix} k_{21} \\ k_{11} \end{matrix}\Big) \overline{F}_m\begin{pmatrix} x_1\cdots x_m \\ y_1\cdots y_m \end{pmatrix}; \begin{matrix} k_{12} \\ k_{22} \end{matrix}\Big) dx_1\cdots dx_m\, dy_1\cdots dy_m\,.$$

IV. $$F_0\begin{bmatrix} k_{11}\, k_{12} \\ k_{21}\, k_{22} \end{bmatrix}=F_0(k_{11})\, F_0(k_{22})+\sum_{m=1}^{\infty}\frac{(-1)^m}{(m!)^2}\int\cdots\int\Big[F_0(k_{11})K_{12}\begin{pmatrix} x_1\cdots x_m \\ y_1\cdots y_m \end{pmatrix}$$

$$+\int\sum_{i=1}^{m} F_1\begin{pmatrix} x_i \\ s_1 \end{pmatrix}; k_{11}\Big) K_{12}\begin{pmatrix} x_1\cdots x_{i-1}\, s_1\, x_{i+1}\cdots x_m \\ y_1\cdots\cdots\cdots y_i\cdots\cdots y_m \end{pmatrix} ds_1+\cdots\cdots$$

$$+\frac{1}{(q!)^2}\int\cdots\int\sum_{i_1=1}^{m}\cdots\sum_{i_q=1}^{m} F_q\begin{pmatrix} x_{i_1}\cdots x_{i_q} \\ s_1\cdots s_q \end{pmatrix}; k_{11}\Big)$$

$$\times K_{12}\begin{pmatrix} x_1\cdots x_{i_1-1}\, s_1\, x_{i_1+1}\cdots x_{i_q-1}\, s_q\, x_{i_q+1}\cdots x_m \\ y_1\cdots\cdots\cdots y_{i_1}\cdots\cdots\cdots\cdots y_{i_q}\cdots\cdots\cdots y_m \end{pmatrix} ds_1\cdots ds_q$$

$$+\cdots\cdots+\frac{1}{m!}\int\cdots\int F_m\begin{pmatrix} x_1\cdots x_m \\ s_1\cdots s_m \end{pmatrix}; k_{11}\Big) K_{12}\begin{pmatrix} s_1\cdots s_m \\ y_1\cdots y_m \end{pmatrix} ds_1\cdots ds_m\Big]$$

$$\times\left[F_0(k_{22})\,K_{21}\begin{pmatrix} y_1\cdots y_m \\ x_1\cdots x_m\end{pmatrix}+\Big\}\sum_{i=1}^{m} F_1\begin{pmatrix} y_i\,;\,k_{22}\\ s_1\end{pmatrix} K_{21}\begin{pmatrix} y_1\cdots y_{i-1}\,s_1\,y_{i+1}\cdots y_m \\ x_1\cdots\cdots x_i\cdots\cdots\cdots x_m\end{pmatrix} ds_1\right.$$

$$+\cdots\cdots+\frac{1}{(q!)^2}\int\cdots\int\sum_{i_1=1}^{m}\cdots\sum_{i_q=1}^{m} F_q\begin{pmatrix} y_{i_1}\cdots y_{i_q}\,;\,k_{22}\\ s_1\cdots s_q\end{pmatrix}$$

$$\times K_{21}\begin{pmatrix} y_1\cdots y_{i_1-1}\,s_1\,y_{i_1+1}\cdots y_{i_q-1}\,s_q\,y_{i_q+1}\cdots y_m \\ x_1\cdots\cdots\cdots x_{i_1}\cdots\cdots\cdots\cdots\cdots x_{i_q}\cdots\cdots\cdots x_m\end{pmatrix} ds_1\cdots ds_q+\cdots\cdots$$

$$\left.+\frac{1}{m!}\int\cdots\int F_m\begin{pmatrix} y_1\cdots y_m\,;\,k_{22}\\ s_1\cdots s_m\end{pmatrix} K_{21}\begin{pmatrix} s_1\cdots s_m \\ x_1\cdots x_m\end{pmatrix} ds_1\cdots ds_m\right] dx_1\cdots$$

$$dx_m\,dy_1\cdots dy_m\,.$$

V. $$F_0\begin{bmatrix} k_{11} & k_{12}\\ k_{21} & k_{22}\end{bmatrix}=F_0(k_{11}+\int k_{12}\,(k_{21}+\int K_{22}\,k_{21}))\;F_0(k_{22}).$$

VI. $$F_0\begin{bmatrix} k_{11} & k_{12}\\ k_{21} & k_{22}\end{bmatrix}=F_0\Big(\int(k_{12}+\int K_{11}\,k_{12})(k_{21}+\int K_{22}\,k_{21})\Big)\;F_0(k_{11})\,F_0(k_{22}).$$

§ 2. Consider the linear integral expression

$$(1)\qquad \varphi_1(x)-\int k_{11}(x,y)\,\varphi_1(y)\,dy-\int k_{12}(xy)\,\varphi_2(y)\,dy=f_1(x).$$

We have defined linear dependence as follows: The linear integral expression (1) is said to be linearly dependent if there exists a set of functions $\varphi_1(x)$, $\varphi_2(x)$, not both zero, which satisfies the system of equations:

$$(2)\qquad \begin{aligned} &\varphi_1(x)-\int k_{11}(x,y)\,\varphi_1(y)\,dy-\int k_{12}(x,y)\,\varphi_2(y)\,dy=0\\ &\varphi_2(x)-\int k_{21}(x,y)\,\varphi_1(y)\,dy-\int k_{22}(x,y)\,\varphi_2(y)\,dx=0\end{aligned}$$

for every $k_{21}(x,y)$ and $k_{22}(x,y)$, i.e. if the Fredholm determinant of the system of equations (2) is zero for all $k_{21}(x,y)$ and $k_{22}(x,y)$.

We then have

Theorem I. A necessary and sufficient condition that (1) be linearly dependent is that $k_{11}(x,y)$ and $k_{11}(x,y)$ be such that

$$F_0(k_{11})=0$$

and

$$\int\cdots\int F_m\begin{pmatrix} x_1\,x_2\cdots x_m\,;\,k_{11}\\ s_1\,s_2\cdots s_m\end{pmatrix} K_{12}\begin{pmatrix} s_1\,s_2\cdots s_m \\ y_1\,y_1\cdots y_m\end{pmatrix} ds_1\,ds_2\cdots ds_m=0$$

for $m=1, 2, 3, \cdots\cdots$ .

For, from formula IV of the preceding section it follows at once that if these conditions are fulfilled, then we must have

$$F_0\begin{bmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{bmatrix}=0$$

for every $k_{21}(x, y)$ and $k_{22}(x, y)$, i.e. our condition is sufficient.

To show that the condition is also necessary let us set

$$k_{21}(x, y)=\lambda\, k_{21}(x, y).$$

Then Formula IV gives us $F_0\begin{bmatrix} k_{11} & k_{12} \\ \lambda k_{21} & k_{22} \end{bmatrix}$ as a power series in $\lambda$ which by hypothesis would be zero for every value of $\lambda$. Hence the coefficient of each power of $\lambda$ will be equal to zero, i.e.

$$F_0(k_{11})\, F_0(k_{22})=0,$$

$$\iint\left[F_0(k_{11})\, k_{12}(x_1\, y_1)+\int F_1\begin{pmatrix} x_1 \\ s_1 \end{pmatrix}; k_{11}\Big) K_{12}\begin{pmatrix} s_1 \\ y_1 \end{pmatrix} ds_1\right]$$

$$\times\left[F_0(k_{22})\, \overline{K}_{21}\begin{pmatrix} y_1 \\ x_1 \end{pmatrix}+\int F_1\begin{pmatrix} y_1 \\ s_1 \end{pmatrix}; k_{22}\Big) \overline{K}_{21}\begin{pmatrix} s_1 \\ x_1 \end{pmatrix} ds_1\right] dx_1\, dy_1=0,$$

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$$\int\cdots\int\Bigg[F_0(k_{11})\, K_{12}\begin{pmatrix} x_1\cdots x_m \\ y_1\cdots y_m \end{pmatrix}+\cdots\cdots+\frac{1}{(q!)^2}$$

$$\int\cdots\cdots\int\sum_{i_1=1}^{m}\cdots\sum_{i_q=1}^{m} F_q\begin{pmatrix} x_{i_1}\cdots x_{i_q} \\ s_1\cdots\cdots s_q \end{pmatrix}; k_{11}\Big)$$

$$\times K_{12}\begin{pmatrix} x_1\cdots x_{i_1-1}\, s_1\, x_{i_1+1}\cdots x_{i_q-1}\, s_q\, x_{i_q+1}\cdots x_m \\ y_1\cdots\cdots\cdots y_{i_1}\cdots\cdots\cdots\cdots\cdots\cdots y_{i_q}\cdots\cdots\cdots y_m \end{pmatrix} ds_1\cdots ds_q+\cdots\cdots$$

$$+\frac{1}{m!}\int\cdots\int F_m\begin{pmatrix} x_1\cdots x_m \\ s_1\cdots s_m \end{pmatrix}; k_{11}\Big) K_{12}\begin{pmatrix} s_1\cdots s_m \\ y_1\cdots y_m \end{pmatrix} ds_1\cdots ds_m\Bigg]$$

$$\times\left[F_0(k_{22})\, \overline{K}_{21}\begin{pmatrix} y_1\cdots y_m \\ x_1\cdots x_m \end{pmatrix}+\cdots+\frac{1}{(q!)^2}\int\cdots\int\sum_{i_1=1}^{m}\cdots\sum_{i_q=1}^{m} F_q\begin{pmatrix} y_{i_1}\cdots y_{i_q} \\ s_1\cdots\cdots s_q \end{pmatrix}; k_{22}\Big)\right.$$

$$\times\overline{K}_{21}\begin{pmatrix} y_1\cdots y_{i_1-1}\, s_1\, y_{i_1+1}\cdots y_{i_q-1}\, s_q\, y_{i_q+1}\cdots y_m \\ x_1\cdots\cdots\cdots x_{i_1}\cdots\cdots\cdots\cdots x_{i_q}\cdots\cdots\cdots x_m \end{pmatrix} ds_1\cdots ds_q+\cdots\cdots$$

$$+\frac{1}{m!}\int\cdots\int F_m\begin{pmatrix} y_1\cdots y_m \\ s_1\cdots y_m \end{pmatrix}; k_{22}\Big)\bar{K}_{21}\begin{pmatrix} s_1\cdots s_m \\ x_1\cdots x_m \end{pmatrix} ds_1\cdots ds_m\Big]$$

$$\times dx_1\cdots\cdots dx_m\, dy_1\cdots\cdots dy_m=0$$

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If in these equations we set $k_{21}(y,x)=k_{12}(x,y)$ and $k_{22}(x,y)=k_{11}(x,y)$, then since from $\int[\varphi(x)]^2\,dx=0$ it follows that $\varphi(x)=0$ we get

$$F_0(k_{11})=0,$$

$$\int F_1\begin{pmatrix} x_1 \\ s_1 \end{pmatrix}; k_{11}\Big) K_{12}\begin{pmatrix} s_1 \\ y_1 \end{pmatrix} ds_1=0,$$

.....

.....

$$\int\cdots\int F_m\begin{pmatrix} x_1\cdots x_m \\ s_1\cdots s_m \end{pmatrix}; k_{11}\Big) K_{12}\begin{pmatrix} s_1\cdots s_m \\ y_1\cdots y_m \end{pmatrix} ds_1\cdots\cdots ds_m=0,$$

.....

.....

which are the conditions of our theorem. From Formula V of the preceding section we obtain a second condition for linear dependence viz:

Theorem II. A necessary and sufficient condition that (1) be linearly dependent is that $k_{11}(x,y)$ and $k_{12}(x,y)$ be such that

$$F_0(k_{11}+\int k_{12}\,k_a)=0$$

for every $k_a(x,y)$.

For we must have

$$F_0\begin{bmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{bmatrix}=F_1\Big(k_{11}+\int k_{12}\,(k_{21}+\int K_{22}\,k_{21})\Big)\,F_0(k_{22})=0$$

for every $k_{21}(x,y)$ and $k_{22}(x,y)$.

Choose $k_{22}(x,y)$ such that $F_0(k_{22})\neq 0$. Then we must have

$$F_0(k_{11}+\int k_{12}(k_{21}+\int K_{22}\,k_{21}))=0$$

i.e. if we determine $k_{21}(x,y)$ so that $k_{21}+\int K_{22}\,k_{21}=k_a$, which is possible

since $F_0(k_{22}) \neq 0$, then we must have $F_0(k_{11} + \int k_{12} k_a) = 0$ for every $k_a(x, y)$. Hence our condition is necessary.

On the other hand, suppose $F_0(k_{11} + \int k_{12} k_a) = 0$ for every $k_a(x, y)$. Then if $F_0(k_{22}) \neq 0$ and we set

$$k_{21}(x, y) = k_{21}(x, y) + \int K_{22}(x, s)\, k_{21}(s, y)\, ds$$

we shall have

$$F_0 \begin{bmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{bmatrix} = 0 .$$

Since $F_0 \begin{bmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{bmatrix}$ is continuous when considered as a function of $k_{22}(x, y)$ it follows that it will be zero for every value of $k_{22}(x, y)$. Hence our condition is also sufficient.

From Formula III′ of §1, it is easy to deduce a simple necessary condition for linear dependence in (1) which is not sufficient. We have

Theorem III. A necessary condition that (1) be linearly dependent is that $F_0(k_{11} + \lambda k_{12}) = 0$ for all values of $\lambda$.

For we must have

$$F_0 \begin{bmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{bmatrix} = F_0(k_{11})\, F_0(k_{22}) + \sum_{i=1}^{m} \frac{(-1)^m}{(m!)^2}$$

$$\times \int \cdots \int \overline{F}_0 \begin{pmatrix} x_1 \cdots x_2 ; & k_{21} \\ y_1 \cdots y_2 & k_{22} \end{pmatrix} \overline{F}_0 \begin{pmatrix} y_1 \cdots y_m ; & k_{12} \\ x_1 \cdots x_m & k_{22} \end{pmatrix} dx_1 \cdots dx_m\, dy_1 \cdots dy_m = 0$$

for every $k_{21}(x, y)$ and $k_{22}(x, y)$. Choose $k_{21}(x, y) = \lambda k_{11}(x, y)$ and $k_{22}(x, y) = \lambda k_{12}(x, y)$. Then we must have

$$F_{22} \begin{bmatrix} k_{11} & k_{12} \\ \lambda k_{11} & \lambda k_{12} \end{bmatrix} = F_0(k_{11})\, F_4(\lambda k_{11}) + \sum_{i=1}^{m} \frac{(-\lambda)^m}{(m!)^2}$$

$$\times \int \cdots \int F_m \begin{pmatrix} x_1 \cdots x_m ; & k_{11} \\ y_1 \cdots y_m & \end{pmatrix} F_m \begin{pmatrix} y_1 \cdots y_m ; & \lambda k_{12} \\ x_1 \cdots x_m & \end{pmatrix} dx_1 \cdots dx_m\, dy_1 \cdots dy_m$$

$$= F_0(k_{11} + \lambda k_{12}) = 0$$

which shows that our condition is necessary[1].

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[1] This theorem can be easily proved directly. For choose $k_{21}(x, y) = \lambda k_{22}(x, y)$ and $k_{22}(x, y) = \lambda k_{12}(x, y)$. Then the existence of a non-zero solution of the system (2) is equivalent to the existence of a non-zero solution of the single equation

$$\varphi_1(x) - \int \Big( k_{11}(x, y) + \lambda k_{22}(x, y) \Big) \varphi_1(y)\, dy = 0$$

i. e. we must have $F_0(k_{11} + \lambda k_{22}) = 0$ for all values of $\lambda$.

On the other hand it is possible to obtain a simple condition for linear dependence which is sufficient without being necessary.

Suppose $F_0(k_{21})=0$ and that $F_0\begin{pmatrix} x_1\cdots x_m \\ y_1\cdots y_m \end{pmatrix}; k_{11}\Big)$ is the first non-zero minor of the Fredholm determinant of $k_{11}(x, y)$. Then any solution $\psi_i(s)$ of the integral equation

$$(3) \qquad \psi(s)-\int\psi(y)\, k_{11}(y, s)\, dy=0$$

will be of the form([1])

$$\psi_i(s)=\sum_{i=1}^{m} a_i\, F_m\begin{pmatrix} x_1^{\circ}\cdots\cdots x_i^{\circ}\cdots\cdots x_m^{\circ} \\ s_1^{\circ}\cdots s_{i-1}^{\circ}\; s\; s_{i+1}^{\circ}\cdots s_m^{\circ} \end{pmatrix}; k_{11}\Big).$$

Assume now that we have

$$\int F_m\begin{pmatrix} x_1\cdots x_m \\ s_1\cdots s_m \end{pmatrix}; k_{11}\Big) K_{12}\begin{pmatrix} s_1 \\ y_1 \end{pmatrix} ds_1=0$$

for all values of $x_i(i=1, 2,\cdots\cdots m)$; $s_j\,(j=2, 3,\cdots\cdots m)$; and $y_1$.

It follows then that

$$\int\cdots\int F_m\begin{pmatrix} x_1\cdots x_m \\ s_1\cdots s_m \end{pmatrix}; k_{11}\Big) K_{12}\begin{pmatrix} s_1\cdots s_m \\ y_1\cdots y_m \end{pmatrix} ds_1\cdots ds_m=0$$

for all values of $x_i$ and $y_i$ $(i=1, 2,\cdots, m)$. This is obvious if $K_{12}\begin{pmatrix} s_1\cdots s_m \\ y_1\cdots y_m \end{pmatrix}$ be expanded according to the minors of the elements of the first column.

Also if we multiply the formula([2])

$$F_{m+1}\begin{pmatrix} x\, x_1\cdots x_m \\ s\; s_1\cdots s_m \end{pmatrix}; k_{11}\Big)-\int k_{11}(t, s)\, F_{m+1}\begin{pmatrix} x\, x_1\cdots x_m \\ t\; s_1\cdots s_m \end{pmatrix}; k_{11}\Big)\, dt$$

$$\equiv k_{11}(x, s)\, F_m\begin{pmatrix} x_1\cdots x_m \\ s_1\cdots s_m \end{pmatrix}; k_{11}\Big)-\sum_{i=1}^{m} k_{11}(x_i\, s)\, F_m\begin{pmatrix} x_1\cdots x_{i-1}\, x\, x_{i+1}\cdots x_m \\ s_1\cdots\cdots\cdots s_i\cdots\cdots s_m \end{pmatrix}; k_{11}\Big)$$

through by $K_{12}\begin{pmatrix} s_1\cdots s_m \\ y_1\cdots y_m \end{pmatrix} ds_1\cdots ds_m$ and integrate it follows that

$$\int\cdots\int F_{m+1}\begin{pmatrix} x\, x_1\cdots x_m \\ s\; s_1\cdots s_m \end{pmatrix}; k_{11}\Big) K_{12}\begin{pmatrix} s_1\cdots s_m \\ y_1\cdots y_m \end{pmatrix} ds_1\cdots ds_m$$

$$-\int k_{11}(t, s)\Big[\int\cdots\int F_{m+1}\begin{pmatrix} x\, x_1\cdots x_m \\ t\; s_1\cdots s_m \end{pmatrix}; k_{11}\Big) K_{12}\begin{pmatrix} s_1\cdots s_m \\ y_1\cdots y_m \end{pmatrix} ds_1\cdots ds_m\Big]\, dt\equiv 0$$

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([1]) C.f. for instance, Heywood and Fréchet, Équation de Fredholm, p. 71.

([2]) C.f. Heywood and Fréchet loc. cit. p. 70.

i.e. $$\int\cdots\int F_{m+1}\begin{pmatrix} x^0\, x_1^0 \cdots\cdots x_m^0 \\ s\ \ s_1 \cdots\cdots s_m \end{pmatrix}; k_{11}\Big) K_{12}\begin{pmatrix} s_1 \cdots\cdots s_m \\ y_1^0 \cdots\cdots y_m^0 \end{pmatrix} ds_1 \cdots\cdots ds_m$$

is a solution of (3). Hence after placing

$$\int\cdots\int F_{m+1}\begin{pmatrix} x\ x_1 \cdots\cdots x_m \\ s\ \ s_1 \cdots\cdots s_m \end{pmatrix}; k_{11}\Big) K_{12}\begin{pmatrix} s_1 \cdots\cdots s_m \\ y_1 \cdots\cdots y_m \end{pmatrix} ds_1 \cdots\cdots ds_m$$

$$=\int\cdots\int F_{m+1}\begin{pmatrix} x_1 x_2 \cdots\cdots x_{m+1} \\ t_1\ \ s_2 \cdots\cdots s_{m+1} \end{pmatrix}; k_{11}\Big) K_{12}\begin{pmatrix} s_2 \cdots\cdots s_{m+1} \\ y_2 \cdots\cdots y_{m+1} \end{pmatrix} ds_2 \cdots\cdots ds_{m+1}$$

we have

$$\int\cdots\int F_{m+1}\begin{pmatrix} x_1 x_2, \cdots\cdots, x_{m+1} \\ t,\ s_2, \cdots\cdots, s_{m+1} \end{pmatrix}; k_{11}\Big) K_{12}\begin{pmatrix} s_2 \cdots\cdots s_{m+1} \\ y_2 \cdots\cdots y_{m+1} \end{pmatrix} ds_2 \cdots\cdots ds_{m+1}$$

$$=\sum_{i=1}^{m} \xi_i\, F_m\begin{pmatrix} x_1 \cdots\cdots x_i \cdots\cdots x_m \\ s_1 \cdots s_{i-1}\, t\, s_{i+1} \cdots s_m \end{pmatrix}; k_{11}\Big)$$

where the functions $\xi_i$ do not contain the variable $t$. Multiplying this equality through by $K_{12}\begin{pmatrix} t \\ y_1 \end{pmatrix} dt$ and integrating we get

$$\int\cdots\int F_{m+1}\begin{pmatrix} x_1, x_2, \cdots\cdots x_{m+1} \\ t\ \ s_2, \cdots\cdots, s_{m+1} \end{pmatrix}; k_{11}\Big)$$

$$\times K_{12}\begin{pmatrix} s_2 \cdots\cdots s_{m+1} \\ y_2 \cdots\cdots y_{m+1} \end{pmatrix} K_{12}\begin{pmatrix} t \\ y_1 \end{pmatrix} ds_2 \cdots\cdots ds_{m+1}\, dt \equiv 0$$

in $x_1, x_2 \cdots\cdots x_{m+1}$; $y_1, y_2 \cdots\cdots y_{m+1}$.

From this it follows that

$$\int\cdots\int F_{m+1}\begin{pmatrix} x_1 x_2 \cdots\cdots x_{m+1} \\ s_1\ s_2 \cdots\cdots s_{m+1} \end{pmatrix}; k_{11}\Big) K_{12}\begin{pmatrix} s_1\ \ s_2 \cdots\cdots s_{m+1} \\ y_1 y_2 \cdots\cdots y_{m+1} \end{pmatrix} ds_1 \cdots\cdots ds_{m+1} \equiv 0$$

in $x_1, x_2 \cdots\cdots, x_{m+1}$, $y_1, y_2 \cdots\cdots y_{m+1}$, which is obvious if $K_{12}\begin{pmatrix} s_1\ s_2 \cdots\cdots s_{m+1} \\ y_1 y_2 \cdots\cdots y_{m+1} \end{pmatrix}$ be expanded according to the elements of the first column.

By continued repetition of the above process we can show that

$$\int\cdots\int F_{m+2}\begin{pmatrix} x_1 \cdots\cdots x_{m+2} \\ s_1 \cdots\cdots s_{m+2} \end{pmatrix}; k_{11}\Big) K_{12}\begin{pmatrix} s_1 \cdots\cdots s_{m+2} \\ y_1 \cdots\cdots y_{m+2} \end{pmatrix} ds_1 \cdots\cdots ds_{m+2} \equiv 0$$

.....

.....

in $x$ and $y$, i.e. we shall have from Formula IV, § 1, $F_0\begin{bmatrix} k_{11}\ k_{12} \\ k_{21}\ k_{22} \end{bmatrix} = 0$

for all values of $k_{21}(x, y)$ and $k_{22}(x, y)$.

We then have proved

Theorem IV. A sufficient condition for linear dependence in (1) is that $k_{11}(x, y)$ and $k_{12}(x, y)$ be such that $F_0(k_{11})=0$ and $\int F_m\left(\begin{matrix} x_1 \cdots\cdots x_m \\ s_1 \cdots\cdots s_m \end{matrix}; k_{11}\right) K_{12}\left(\begin{matrix} s_1 \\ y_1 \end{matrix}\right) ds_1 = 0$ where $F_m\left(\begin{matrix} x_1 \cdots\cdots x_m \\ s_1 \cdots\cdots s_m \end{matrix}; k_{11}\right)$ is the first minor of the Fredholm determinant of $k_{11}(x, y)$ which is not identically zero.

§ 3. At the end of § 2 we found that if $F_0(k_{11})=0$ and $F_m\left(\begin{matrix} x_1 \cdots\cdots x_m \\ y_1 \cdots\cdots y_m \end{matrix}; k_{11}\right)$ is the first minor of the Fredholm determinant of $k_{11}(x, y)$ which is not identically zero, then $F_0(k_{11})=0$ and $\int F_m\left(\begin{matrix} x_1 \cdots\cdots x_m \\ s_1 \cdots\cdots s_m \end{matrix}; k_{11}\right) k_{12}(s_1 y_1)\, ds_1 = 0$ for every $x_1, \cdots\cdots, x_m$; $s_2 \cdots\cdots s_m$; and $y_1$, are sufficient conditions for linear dependence in (1). If we recall that for any set of values $x_1^0, \cdots\cdots, x_m^0$; $y_1^0 \cdots\cdots y_{i-1}^0\ y_{i+1}^0 \cdots\cdots y_m^0$, $F_m\left(\begin{matrix} x_1^0 \cdots\cdots\ x_i^0\ \cdots\cdots x_m^0 \\ y_1^0 \cdots\ y_{i-1}^0\ y\ y_{i+1}^0 \cdots y_m^0 \end{matrix}; k_{11}\right)$ is a solution of the homogeneous equation

$$\psi_1(y) - \int \psi_1(x)\ k_{11}(x, y)\, dx = 0,$$

this result suggests the question whether the existence of a solution of the two equations

$$(4) \qquad \begin{aligned} \psi_1(x) - \int \psi_1(y)\ k_{11}(y, x)\, dy &= 0, \\ \int \psi_1(y)\ k_{12}(y, x)\, dy &= 0, \end{aligned}$$

is not a sufficient and even a necessary condition for linear dependence in (1). It turns out that it is both necessary and sufficient, i.e. we have a theorem analogous to Theorm I of the introduction, viz:

Theorem I. A necessary and sufficient condition that (1) be linearly dependent is that there exists a function $\psi_1(x)$ not zero such that the relations (4) are satisfied.

Let us assume that $\psi_1(x) = \psi_1^0(x)$ is a common solution of the equations (4). Then if we choose this $\psi_1^0(x)$ together with $\psi_2(x)=0$ we shall have for every $k_{21}(x, y)$ and $k_{22}(x, y)$ a non-zero solution of the system

$$(5)\qquad \begin{aligned} &\psi_1(x)-\int\psi_1(y)\,k_{11}(y,x)\,dy-\int\psi_2(y)\,k_{21}(y,x)\,dy=0,\\ &\psi_2(x)-\int\psi_1(y)\,k_{12}(y,x)\,dy-\int\psi_2(y)\,k_{22}(y,x)\,dy=0, \end{aligned}$$

which is the homogeneous adjoint system of (2). But the existence of a non-zero solution of (5) implies the existence of a non-zero solution of (2). The expression (1) is then linearly dependent and we have shown that the condition stated in our theorem is sufficient.

On the other hand, the condition is also necessary, i. e. there exists a solution of the system of equations (4) which is not identically zero.

Let us first show that we must have $F_0(k_{11})=0$. Since $k_{21}(x,y)$ and $k_{22}(x,y)$ are arbitrary let us choose $k_{21}(x,y)=0$. Now if the system (2) has a solution other than $\varphi_1(x)=\varphi_2(x)=0$, it is also true of the system (5). Then the first equation of (5) gives

$$(6)\qquad \psi_1(x)-\int\psi_1(y)\,k_{11}(y,x)\,dy=0.$$

There are now two cases to be considered :

(a) $\qquad F_0(k_{11})\neq 0,$

(b) $\qquad F_0(k_{11})=0.$

Suppose that (a) is true. Then the only possible solution of the equation (6) would be $\psi_1(x)=0$ and the second equation of (5) would give

$$\psi_2(x)-\int\psi_2(y)\,k_{22}(y,x)\,dy=0.$$

Consequently, since by hypothesis there exists a non-zero solution of (5), it would be necessary to have $F_0(k_{22})=0$, for every $k_{22}(x,y)$, which is impossible, i.e. $k_{11}(x,y)$ must be such that $F_0(k_{11})=0$ and there exist solutions of the equation (6).

Now let us show that some solution of equation (6) satisfies also the second of the equations (4).

Suppose that the $m$ functions $\psi_1^{(1)}(x), \psi_1^{(2)}(x), \cdots\cdots, \psi_1^{(m)}(x)$ constitute a complete system of linearly independent solutions of the equation (6) and let us substitute there $\psi_1^{(i)}(y)$, $(i=1,2,\cdots\cdots,m)$ for $\psi_1(y)$ in the second equation of (4) and place

$$\psi^{(i)}(x)=\int\psi_1^{(i)}(y)\,k_{12}(y,x)\,dy \qquad i=1,2,\cdots\cdots,m.$$

Then either

(a) the $\varphi_2^{(i)}(x)$ are linearly independent,

or

(b) the $\varphi_2^{(i)}(x)$ are linearly dependent.

If the $\varphi_2^{(i)}(x)$ are linearly independent, there exists an equivalen-set of functions $\xi^{(1)}(x), \xi^{(2)}(x), \cdots, \xi^{(m)}(x)$, which are normed and ort thogonal and such that the $\xi^{(i)}(x)$ are linear combinations of the $\xi_2^{(i)}(x)$ and conversely, the $\varphi_2^{(i)}(x)$ are linear combinations of the $\xi^{(i)}(x)$.

If now the $m$ normed and orthogonal functions $\varphi_1^{(1)}(x), \varphi_2^{(2)}(x), \cdots\cdots, \varphi_m^{(m)}(x)$, constitute a complete system of linear independent solutions of the equation

$$\varphi_1(x) - \int k_{11}(x, y)\, \varphi_1(y)\, dy = 0, \tag{7}$$

let us choose([1])

$$k_{21}(x, y) = \sum_{i=1}^{m} \xi^{(i)}(x)\, \varphi_1^{(i)}(y)$$

and

$$k_{22}(x, y) = \sum_{i=1}^{m} \xi^{(i)}(x)\, \xi^{(i)}(y).$$

Then obviously the $\varphi_2^{(i)}(x)$ will constitute a complete linearly independent set of solutions of the equation

$$\varphi_2(x) - \int k_{22}(x, y)\, \varphi_2(y)\, dy = 0. \tag{8}$$

Then, if the system

$$\begin{aligned} &\psi_1(x) - \int \psi_1(y)\, k_{11}(y, x)\, dy - \int \psi_2(y) \sum_{i=1}^{m} \xi^{(i)}(y)\, \varphi_1^{(i)}(x)\, dy = 0, \\ &\psi_2(x) - \int \psi_1(y)\, k_{12}(y, x)\, dy - \int \psi_2(y) \sum_{i=1}^{m} \xi^{(i)}(y)\, \xi^{(i)}(x)\, dy = 0 \end{aligned} \tag{5'}$$

is to have a solution not identically zero, it must be possible to solve the first equation for $\psi_1(x)$ in terms of $\psi_2(x)$ and then substitute this value of $\psi_1(x)$ in the second equation and solve for $\psi_2(x)$.

Since $F_0(k_{11}) = 0$, it is possible to find a solution of the first equation of (5′) only if

$$\int \left[ \int \psi_2(y) \sum_{i=1}^{m} \xi^{(i)}(y)\, \varphi_1^{(i)}(x)\, dy \right] \varphi_1^{(l)}(x)\, dx = 0$$

$$l = 1, 2, \cdots\cdots, m.$$

([1]) Cf. Schmidt, Mathematische Annalen, vol. 63 (1907) p. 449.

But since the $\varphi_1^{(i)}(x)$ are normed and orthogonal functions we must have

$$(9) \qquad \int \psi_2(y)\, \xi^{(i)}(y)\, dy = 0.$$

The solution of the first equation of (5′) is then

$$\psi_1(x) = \int \psi_2(y) \sum_{i=1}^{m} \xi^{(i)}(y)\, \varphi_1^{(i)}(x)\, dy + \int \left[ \int \psi_2(z) \sum_{i=1}^{m} \xi^{(i)}(z)\, \varphi_1^{(i)}(y)\, dz \right]$$
$$\times L(y, x\,;\, k_{11})\, dy + \sum_{i=1}^{m} a_i\, \psi_1^{(i)}(x)$$

where $L(y, x\,;\, k_{11})$ is the pseudo-reciprocal kernel of $k_{11}(x, y)$ and the $\psi_1^{(i)}(x)$ are the solutions of (6).

Taking (9) into consideration, this value of $\psi_1(x)$ becomes

$$\psi_1(x) = \sum_{i=1}^{m} a_i\, \varphi_1^{(i)}(x).$$

Substituting then, this value of $\psi_1(x)$ in the second equation of (5′) we have

$$\psi_2(x) - \int \sum_{i=1}^{m} a_i\, \psi_1^{(i)}(y)\, k_{12}(y, x)\, dy - \int \psi_2(y)\, k_{22}(y, x)\, dy = 0.$$

Since the Fredholm determinant of $k_{22}(x, y)$ is evidently zero, the condition that this equation have a solution is that the relation

$$10. \qquad \int \left[ \int \sum_{i=1}^{m} a_i\, \psi_1^{(i)}(y)\, k_{12}(y, x)\, dy \right] \varphi_2^{(l)}(x)\, dx = 0$$
$$l = 1, 2, \cdots\cdots, m$$

shall be satisfied, the $\varphi_2^{(l)}(x)$ being the non-zero solutions of (8).

But

$$\varphi_2^{(l)}(x) = \int \psi_1^{(l)}(y)\, k_{12}(y, x)\, dy \qquad l = 1, 2, \cdots\cdots, m.$$

Multiplying (10) through by $a_l$ and summing as to $l$ we get

$$\int \left[ \sum_{i=1}^{m} a_i\, \varphi_2^{(i)}(x) \right] \left[ \sum_{l=1}^{m} a_l\, \varphi_2^{(l)}(x) \right] dx = \int \left[ \sum_{i=1}^{m} a_i\, \varphi_2^{(i)}(x) \right]^2 dx = 0$$

and hence $\sum_{i=1}^{m} a_i\, \varphi_2^{(i)}(x) = 0$, i.e., the $\varphi_2^{(i)}(x)$ being linearly independent, we must have $a_i = 0$ and consequently $\psi_1(x) = 0$.

The second equation of (5′), then, becomes

$$\psi_2(x) - \int \sum_{i=1}^{m} [\psi_2(y)\ \xi^{(i)}(y)\, dy]\ \xi^{(i)}(x) = 0$$

and hence by (9), we must have $\psi_2(x) = 0$.

If then the $\varphi_2^{(i)}(x)$ are linearly independent, the only possible solution of the system (5′) is $\psi_1(x) = \psi_2(x) = 0$ which is contrary to the hypothesis. Hence the condition (b) must be fulfilled, i.e. there exists a set of constants $a_i$ not all zero such that

$$\sum_{i=1}^{m} a_i\, \varphi_2^{(i)}(x) = \sum_{i=1}^{m} a_i \int \psi_1^{(i)}(y)\, k_{12}(y, x)\, dy = \int \sum_{i=1}^{m} a_i\, \psi_1^{(i)}(y)\, k_{12}(y, x)\, dy = 0.$$

But since $\sum_{i=1}^{m} a_i\, \psi_1^{(i)}(x)$ is a solution of (6) we have shown that in order that (1) be linearly dependent it is necessary that the equations (4) have a common solution.

Finally with the aid of the theorem just proved, we shall prove the following

Theorem II. A necessary and sufficient condition that (1) be linearly dependent is that there exists a function $\psi(x)$ such that $\int \psi(x) f_1(x)\, dx = 0$ for every $\psi_1(x)$ and $\psi_2(x)$.

Suppose (1) is linearly dependent. Then by Theorem I there exists a function $\psi_1(x)$ which satisfies the relations (4). Multiplying (1) through by this $\psi_1(x)$ and integrating we get

$$\begin{aligned}\int \psi_1(x) f_1(x)\, dx &= \int \psi_1(x)\, \varphi_1(x)\, dx - \iint \psi_1(x)\, k_{11}(x, y)\, \varphi_1(y)\, dy\, dx \\ &\qquad - \iint \psi_1(x)\, k_{12}(x, y)\, \varphi_2(y)\, dy\, dx \\ &= \int \left[\psi_1(x) - \int \psi_1(y)\, k_{11}(y, x)\, dy\right] \varphi_1(x)\, dx \\ &\qquad - \int \left[\int \psi_1(y)\, k_{12}(y, x)\, dy\right] \varphi_2(x)\, dx.\end{aligned}$$

But by (4) the right hand member of this last identity is zero. Hence the left hand member must also be zero, i. e. we must have

$$\int \psi_1(x) f_1(x)\, dx = 0$$

and we have shown that the condition of our theorem is necessary.

To prove that the condition is also sufficient let us assume the

existence of a function $\psi_1(x)$ such that $\int \psi_1(x) f_1(x)\,dx=0$ for every $\varphi_1(x)$ and $\varphi_2(x)$. Then if we multiply (1) through by this $\psi_1(x)$ and integrate, we shall have $\int\left[\psi_1(x)-\int\psi_1(y)\,k_{11}(y,x)\,dy\right]\varphi_1(x)\,dx$ $-\int\left[\int\psi_1(y)\,k_{12}(y,x)\,dy\right]\varphi_2(x)\,dx=0$ for every $\varphi_1(x)$ and $\varphi_2(x)$.

For

$$\varphi_1(x)=\psi_1(x)-\int\psi_1(y)\,k_{11}(y,x)\,dy \text{ and } \varphi_2(x)=-\int\psi_1(y)\,k_{12}(x,y)\,dy$$

this identity becomes

$$\int\left[\psi_1(x)-\int\psi_1(y)\,k_{11}(y,x)\,dy\right]^2 dx+\int\left[\int\psi_1(y)\,k_{12}(y,x)\,dy\right]^2 dx=0$$

i. e. the function $\psi_1(x)$ must satisfy the relations

$$\psi_1(x)-\int\psi_1(y)\,k_{11}(y,x)\,dy=0,$$

$$\int\psi_1(y)\,k_{12}(y,x)\,dy=0.$$

But by Theorem I this condition is sufficient for linear dependence in (1). It will be noticed that the condition contained in the theorem just proved is analogous to the condition for linear dependence which one finds in connection with the theory of algebraic equations.

## PART II.

Consider now the system of linear integral expressions

$$(1)\qquad \varphi_i(x)-\sum_{j=1}^{n}\int k_{ij}(x,y)\,\varphi_j(y)\,dy=f_i(x) \qquad i=1,2,3,\cdots\cdots,m.$$

This system is said to be linearly dependent if there exists a non-zero solution of the system,

$$(2)\qquad \begin{aligned} &\varphi_i(x)-\sum_{j=1}^{n}\int k_{ij}(x,y)\,\varphi_j(y)\,dy=0 \\ &\qquad\qquad\qquad\qquad\qquad\qquad i=1,2,3,\cdots\cdots,m \\ &\varphi_h(x)-\sum_{j=1}^{n}\int k_{hj}(x,y)\,\varphi_j(y)\,dy=0 \\ &\qquad\qquad\qquad\qquad\qquad\qquad h=m+1,\ m+2,\cdots\cdots,n. \end{aligned}$$

whatever be the functions $k_{h,j}(x, y)$ in the last $n-m$ equations.

It is possible to obtain a condition for linear dependence in (1) in terms of the Fredholm determinants and their minors for the system, but the results are rather complicated. However, the theorems of Part I, § 3 generalize themselves very readily and the demonstrations are closely analogous, so we shall confine ourselves to these in this section.

We have then relative to the system (1)

Theorem I. A necessary and sufficient condition that (1) be linearly dependent is that there exists a set of $m$ functions $\psi_1(x)$, $\psi_2(x)$, $\cdots\cdots$, $\psi_m(x)$, not all zero which satisfy the relations

$$(3)\qquad \begin{aligned} &\psi_i(x)-\sum_{l=1}^{m}\int\psi_l(y)\,k_{li}(y,x)\,dy=0 \\ &\qquad\qquad\qquad\qquad\qquad\qquad i=1,2,3,\cdots\cdots,m. \\ &\sum_{l=1}^{m}\int\psi_l(y)\,k_{lh}(y,x)\,dy=0 \\ &\qquad\qquad\qquad\qquad\qquad\qquad h=m+1,m+2,\cdots\cdots,n. \end{aligned}$$

Suppose there exists a set of functions $\psi_1^{(0)}(x), \psi_2^{(0)}(x), \cdots\cdots, \psi_m^{(0)}(x)$, not all of which are identically zero which satisfy the relations (3). Then these $\psi_i^{(0)}(x)$ together with $\psi_h^{(0)}(x)=0$ $(h=m+1, m+2, \cdots\cdots, n)$ would obviously constitute a non-zero solution of the system

$$(4)\qquad \begin{aligned} &\psi_i(x)-\sum_{j=1}^{n}\int\psi_j(y)\,k_{ji}(y,x)\,dy=0 \\ &\qquad\qquad\qquad\qquad\qquad\qquad i=1,2,\cdots\cdots,m, \\ &\psi_h(x)-\sum_{j=1}^{n}\int\psi_j(y)\,k_{jh}(y,x)\,dy=0 \\ &\qquad\qquad\qquad\qquad\qquad\qquad h=m+1,m+2,\cdots\cdots,n, \end{aligned}$$

which is the homogeneous adjoint system of (2).

But the existence of a non-zero solution of (4) implies the existence of a non-zero solution of (2). The expressions (1) are then linearly dependent and we have shown that the condition of our theorem is sufficient.

To show that the condition is also necessary, i. e. there exists a solution of the equations (3), let us assume that for every $k_{hj}(x, y)$ $(h=m+1, m+2, \cdots\cdots, n\,;\ j=1, 2, \cdots\cdots, n)$ there exists a set of functions $\varphi_1(x), \varphi_2(x), \cdots\cdots, \varphi_n(x)$, not all of which are zero, which satisfies the system (2). Then for every $k_{hj}(x, y)$ there will also exist a set of functions $\psi_1(x), \psi_2(x), \cdots\cdots, \psi_n(x)$, not all zero, which satisfies the system (4).

First let us show that we must have

$$F_0[k_{li}]=0 \qquad l, i=1, 2, \cdots\cdots, m$$

where $F_0[k_{li}]$ denotes the Fredholm determinant of the system of equations whose functions are the $k_{li}(x, y)$ $(l, i,=1, 2, \cdots\cdots m)$.

Since the $k_{h,j}(x, y)$ are arbitrary let us choose $k_{hi}(x, y)=0$ $(h=m+1, m+2, \cdots\cdots, n;\ i=1, 2, \cdots\cdots, m)$. Then from the first $m$ equations of (4) we have

$$(5) \qquad \psi_i(x)-\sum_{l=1}^{m}\int \psi_l(y)\, k_{li}(y, x)\, dy=0 \qquad i=1, 2, \cdots\cdots, m.$$

There are now two cases to be considered:

$$\begin{array}{ll} \text{(a)} & F_0[k_{li}]\neq 0, \\ \text{(b)} & F_0[k_{li}]=0. \end{array} \qquad l, i=1, 2, \cdots\cdots, m.$$

If (a) is true the only possible solutions of (5) would be $\psi_1(x)=\psi_2(x)=\cdots\cdots=\psi_m(x)=0$ and the last $n-m$ equations of (4) would give

$$\psi_h(x)-\sum_{p=m+1}^{n}\int \psi_p(y)\, k_{ph}(y, x)\, dy=0$$

$$h=m+1, m+2, \cdots\cdots, n.$$

Then in order to have a non-zero solution of (4) it would be necessary to have $F_0[k_{ph}]=0$ for all $k_{ph}(x, y)$, $(p, h=m+1, m+2, \cdots\cdots, n)$ which is impossible. We then must have $F_0[k_{li}]=0$ $(l, i=1, 2, \cdots\cdots, m)$, i. e. (b) must be true, and there exist solutions of the system (5) and also solutions of the system

$$(6) \qquad \varphi_i(x)-\sum_{l=1}^{m}\int k_{li}(x, y)\, \varphi_i(y)\, dy=0 \qquad i=1, 2, \cdots\cdots, m.$$

Now let us show that some solution of the first $m$ of the equations (3) satisfies also the last $n-m$.

Suppose the $r$ normed and orthogonal sets of functions $\varphi_1^{(q)}(x), \varphi_2^{(q)}(x), \cdots\cdots, \varphi_m^{(q)}(x)$, $(q=1, 2, \cdots\cdots, r)$ constitute a complete linearly independent set of solutions of the system (6). Suppose also the $r$ sets of functions $\psi_1^{(q)}(x), \psi_2^{(q)}(x), \cdots\cdots, \psi_m^{(q)}(x)$, $(q=1, 2, \cdots\cdots, r)$ constitute a complete linearly independent set of solutions of the system (5). Let us substitute these $\psi_i^{(q)}(x)$ for $\psi_i(x)$ in the last $n-m$ equations of (4) and place

$$\varphi_h^{(q)}(x)=\sum_{i=1}^{m}\int \psi_i^{(q)}(y)\, k_{ih}(y, x)\, dy \qquad \begin{array}{l} h=m+1, m+2, \cdots\cdots, n, \\ q=1, 2, \quad \cdots\cdots\cdots\cdots, m. \end{array}$$

Suppose, if possible, that these $r$ sets of functions $\varphi_h^{(q)}(x)$, $(q=1, 2, \cdots\cdots, r)$ are linearly independent. Then there exist $r$ equivalent sets of functions $\xi_{m+1}^{(q)}(x)$, $\xi_{m+2}^{(q)}(x)$, $\cdots\cdots$, $\xi_n^{(q)}(x)$ such that

$$\sum_{p=m+1}^{n} \int \xi_p^{(q)}(x)\, \xi_p^{(s)}(x)\, dx = \begin{cases} 0 & \text{for } q \neq s \\ 1 & \text{for } q = s \end{cases}$$

and such that

$$\xi_h^{(q)}(x) = \sum_{q=1}^{r} c_q\, \varphi_h^{(q)}(x) \qquad h=m+1, m+2, \cdots\cdots, n.$$

Conversely, the $\varphi_h^{(q)}(x)$ are linear combinations of the $\xi_h^{(q)}(x)$.

Let us then choose $k_{hi}(x, y)$ $\begin{pmatrix} h=m+1, m+2, \cdots\cdots, n \\ i= \quad 1, \quad 2, \quad \cdots\cdots, m \end{pmatrix}$ such that

$$k_{hi}(x, y) = \sum_{q=1}^{r} \xi_h^{(q)}(x)\, \varphi_i^{(q)}(y)$$
$$h=m+1, m+2, \cdots\cdots, n,$$
$$i=1, 2, \cdots\cdots, m,$$

and $k_{hp}(x, y)$ $(h, p=m+1, m+2, \cdots\cdots, n)$ such that

$$k_{hp}(x, y) = \sum_{q=1}^{r} \xi_h^{(q)}(x)\, \xi_p^{(q)}(y)$$
$$h, p=m+1, m+2, \cdots\cdots, n.$$

The Fredholm determinant of the system $k_{h,p}(x, y)$ $(h, p=m+1, m+2, \cdots\cdots, n)$ will then evidently be zero, and the functions $\varphi_h^{(q)}(x)$ will constitute a complete linearly independent set of solutions of the system

$$(7) \qquad \varphi_h(x) - \sum_{p=m+1}^{n} \int k_{h,p}(x, y)\, \varphi_p(y)\, dy = 0$$
$$h=m+1, m+2, \cdots\cdots, n.$$

If then the system

$$(4') \qquad \begin{aligned} &\psi_i(x) - \sum_{l=1}^{m} \int \psi_l(y)\, k_{li}(y, x)\, dy - \sum_{h=m+1}^{n} \int \psi_h(y) \sum_{q=1}^{r} \xi_h^{(q)}(y)\, \varphi_i^{(q)}(x)\, dy = 0 \\ &\qquad\qquad i=1, 2, \cdots\cdots, m \\ &\psi_p(x) - \sum_{l=1}^{m} \int \psi_l(y)\, k_{lp}(y, x)\, dy - \sum_{h=m+1}^{n} \int \psi_h(y) \sum_{q=1}^{r} \xi_h^{(q)}(y)\, \xi_p^{(q)}(x)\, dy = 0 \\ &\qquad\qquad p=m+1, m+2, \cdots\cdots, n, \end{aligned}$$

is to have a solution not identically zero, it must be possible to solve the first $m$ equations for $\psi_i(x)$ $(i=1, 2, \cdots\cdots, m)$ in terms of the $\psi_h(x)$

$(h=m+1, m+2, \cdots\cdots, n)$ and then substitute these values of $\psi_i(x)$ in the last $(n-m)$ equations and solve for $\psi_h(x)$.

Since $$F_0[k_{li}]=0 \qquad l, i=1, 2, \cdots\cdots, m,$$

it is possible to find a solution of the first $m$ equations of (4′) only if

$$\int \sum_{i=1}^{m} \left[ \sum_{h=m+1}^{n} \int \psi_h(y) \sum_{q=1}^{r} \xi_h^{(q)}(y) \varphi_i^{(q)}(x)\, dy \right] \varphi_i^{(q)}(x)\, dx=0$$

$$s=1, 2, \cdots\cdots, r.$$

But since the $\varphi_i^{(q)}(x)$ are normed and orthogonal sets of functions we must have

$$(8) \qquad \sum_{h=m+1}^{n} \int \psi_h(y) \sum_{q=1}^{r} \xi_h^{(q)}(y)\, dy=0.$$

The solution of the first $m$ equations of (4′) is then

$$\psi_i(x)=\sum_{q=1}^{r} a_q \varphi_i^{(q)}(x) \qquad i=1, 2, \cdots\cdots, m,$$

where the $\varphi_i^{(q)}(x)$ are the solutions of the system (5).

Substituting these values of $\psi_i(x)$ in the last $(n-m)$ equations of (4′) we get

$$(9) \qquad \psi_h(x)-\sum_{p=m+1}^{n} \int \psi_p(y)\, k_{ph}(y, x)\, dy=\sum_{i=1}^{m} \int \sum_{q=1}^{r} a_q \varphi_i^{(q)}(y)\, k_{ih}(y, x)\, dy$$

$$h=m+1, m+2, \cdots\cdots, n.$$

Since the Fredholm determinant of $k_{hp}(x, y)$ $(h, p=m+1, m+2, \cdots\cdots, n)$ is obviously zero and since the $\varphi_h^{(q)}(x)$ are the solutions of the homogeneous system adjoint to the system (9), the condition for a solution of (9) is

$$(10) \qquad \int \sum_{h=m+1}^{n} \left[ \sum_{q=1}^{r} a_q \varphi_h^{(q)}(x) \right] \varphi_h^{(s)}(x)\, dx=0 \quad s=1, 2, \cdots\cdots, r.$$

Multiplying (10) through by $a_s$ and summing as to $s$ we get

$$\sum_{s=1}^{r} a_s \int \sum_{h=m+1}^{n} \left[ \sum_{q=1}^{r} a_q \varphi_h^{(q)}(x) \right] \varphi_h^{(s)}(x)\, dx=\int \sum_{h=m+1}^{n} \left[ \sum_{q=1}^{r} a_q \varphi_h^{(q)}(x) \right]^2 dx$$

$$=\sum_{h=m+1}^{n} \int \left[ \sum_{q=1}^{r} a_q \varphi_h^{(q)}(x) \right]^2 dx=0$$

and hence

$$\sum_{q=1}^{r} a_q \varphi_h^{(q)}(x)=0 \qquad h=m+1, m+2, \cdots\cdots, n.$$

i. e., the $\varphi_h^{(q)}(x)$ being linearly independent sets of functions, we must have $a_q=0$ $(q=1, 2, \cdots\cdots, r)$ and consequently $\psi_i(x)=0$ $(i=1, 2, \cdots\cdots, m)$.

The last $(n-m)$ equations of (4′) then become

$$\psi_p(x)-\sum_{h=m+1}^{n}\int\psi_h(y)\sum_{q=1}^{r}\xi_h^{(q)}(y)\,\xi_p^{(q)}(x)\,dy=0$$

and hence by (8) we must have $\psi_p(x)=0$ $(p=m+1, m+2, \cdots\cdots, n)$.

If then, the sets of functions $\varphi_h^{(q)}(x)$ are linearly independent the only possible solution of the system (4′) is $\psi_1(x)=\cdots\cdots=\psi_n(x)=0$ which is contrary to the hypothesis. Hence the $\varphi_h^{(p)}(x)$ must be linearly dependent, i. e. there exists a set of constants $a$ such that

$$\sum_{q=1}^{r}a_q\,\varphi_h^{(q)}(x)=\sum_{q=1}^{r}a_q\sum_{i=1}^{m}\int\psi_i^{(q)}(x)\,k_{ih}(y, x)\,dy$$

$$=\sum_{q=1}^{m}\int\sum_{q=1}^{r}a_q\,\psi_i^{(q)}(x)\,k_{ih}(y, x)\,dy=0.$$

But since $\sum_{q=1}^{r}a_q\,\psi_i^{(q)}(x)$ $(i=1, 2, \cdots\cdots, m)$ is a solution of (5) we have shown that for the expressions (1) to be linearly dependent it is necessary that there exists a non-zero solution of the equations (3).

Relative to the system (1) we have also the following:

Theorem II. A necessary and sufficient condition that the linear integral expressions (1) be linearly dependent is that there exists a set of functions $\psi_1(x), \psi_2(x), \cdots\cdots, \psi_m(x)$ not all zero such that $\sum_{i=1}^{m}\int\psi_i(x)\,f_i(x)\,dx=0$ for every $\varphi_1(x)$, $\varphi_2(x)$, $\cdots\cdots$, $\varphi_n(x)$.

This Theorem is easily proved by the method adopted in proving its analogue in the case of a single linear integral expression, i. e. Theorem II of Part I, § 3. For, suppose the expressions (1) are linearly dependent. Then by Theorem I there exists a set of functions $\psi_1(x)$, $\psi_2(x)$, $\cdots\cdots$, $\psi_m(x)$, which satisfy the relations (3). If we multiply the expressions (1) then by these $\psi_i(x)$, integrate, and add, we get

$$\sum_{i=1}^{m}\int\left(\psi_i(x)-\sum_{l=1}^{m}\int\psi_l(y)\,k_{li}(y, x)\,dy\right)\varphi_i(x)\,dx-\sum_{h=m+1}^{n}\int\left[\sum_{i=1}^{m}\int\psi_i(y)\right.$$

$$\left.\times k_{ih}(y, x)\,dy\right]\varphi_h(x)\,dy=\sum_{i=1}^{m}\int\psi_i(x)\,f_i(x)\,dx.$$

But the left hand member of this identity is zero, hence the right

hand member must also be zero i. e. we must have $\sum_{i=1}^{m}\int \psi_i(x) f_i(x)\,dx=0$ and we have shown that our condition is necessary.

On the other hand, the condition is sufficient. For if there exists a set of functions $\psi_1(x), \psi_2(x), \cdots\cdots, \psi_m(x)$ such that $\sum_{i=1}^{m}\int \psi_i(x) f_i(x)\,dx=0$, and if we multiply the expressions (1) then by these $\psi_i(x)$, integrate and add we shall have

$$\sum_{i=1}^{m}\int\left(\psi_i(x)-\sum_{l=1}^{m}\int \psi_l(y)\,k_{li}(y,x)\,dy\right)\varphi_i(x)\,dx$$
$$-\sum_{i=1}^{m}\int\sum_{h=m+1}^{n}\left(\int \psi_i(x)\,k_{ih}(y,x)\,dy\right)\varphi_h(x)\,dx=0$$

for every $\varphi_1(x), \varphi_2(x), \cdots\cdots, \varphi_n(x)$. Then for

$$\varphi_i(x)=\psi_i(x)-\sum_{l=1}^{m}\int \psi_l(y)\,k_{li}(y,x)\,dy \qquad i=1,2,3,\cdots\cdots,m,$$

$$\varphi_h(x)=-\sum_{i=1}^{m}\int \varphi_i(x)\,k_{ih}(y,x)\,dy \qquad h=m+1,m+2,\cdots\cdots,n,$$

the above identity becomes

$$\sum_{i=1}^{m}\int\left[\psi_i(x)-\sum_{l=1}^{m}\int \psi_l(x)\,k_{li}(y,x)\,dy\right]^2 dx$$
$$+\sum_{h=m+1}^{m}\int\left[\sum_{i=1}^{m}\int \psi_i(x)\,k_{ih}(y,x)\,dy\right]^2 dy=0$$

i. e we must have

$$\psi_i(x)-\sum_{l=1}^{m}\int \psi_l(y)\,k_{li}(y,x)\,dy=0 \qquad i=1,2,3,\cdots\cdots,m,$$

$$\sum_{i=1}^{m}\int \psi_l(y)\,k_{ih}(y,x)\,dy=0 \qquad h=m+1,m+2,\cdots\cdots,n.$$

But by Theorem I, this is a sufficient condition for linear dependence in (1).

We have thus shown that in the definition of linear dependence for a system of linear integral expressions:

$$\varphi_i(x) - \sum_{j=1}^{n} \int k_{ij}(x,y)\, \varphi_j(x)\, dx = f_i(x) \qquad i=1,2,3,\cdots\cdots,m,$$

we can proceed in three equivalent ways analogous to the situation for algebraic equations viz:

(a) for every $k_{h,j}(x,y)$, $(h=m+1, m+2, \cdots\cdots n;\ j=1, 2, 3, \cdots\cdots n)$ the Fredholm determinant of the system

$$\varphi_i(x) - \sum_{j=1}^{n} \int k_{ij}(x,y)\, \varphi_j(y)\, dy = 0 \qquad i=1, 2, \cdots\cdots, m,$$

$$\varphi_h(x) - \sum_{j=1}^{n} \int k_{hj}(x,y)\, \varphi_j(y)\, dy = 0 \qquad h=m+1, m+2, \cdots\cdots, n,$$

is identically zero;

(b) there exists at least one solution of the adjoint system,

$$\psi_i(x) - \sum_{l=1}^{m} \int \psi_l(y)\, k_{li}(y,x)\, dy = 0 \qquad i=1, 2, \cdots\cdots, m,$$

$$\sum_{l=1}^{m} \int \psi_l(y)\, k_{lh}(y\ x)\, dy = 0 \qquad h=m+1, m+2, \cdots\cdots, n,$$

and

(c) there exists a set of functions $\psi_i(x)$, $(i=1, 2, 3, \cdots\cdots, m)$ such that

$$\sum_{i=1}^{m} \int f_i(x)\, \psi_i(x)\, dx = 0$$

for every function $\psi_j(x)$ $(j=1, 2, 3, \cdots\cdots, n)$.

Finally we might note that the above theory is easily extensible in the sense of E. H. Moore's general integral equation theory([1]).

Ann Arbor, Mich, June 10th, 1918.

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([1]) Bulletin of American Mathematical Society, vol. 18 (1911–12), pp. 334–362.

# 抄　錄　短　評

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Segre, Sulla geometria delle schiere rigate o regoli, e in particolare sui complessi lineari di tali enti. Bianchi, Le trasformazioni di Ribaucour dei sistemi *n pli* ortogonali e il teorema generale di permutabilità. Nencini, Sulla classificazione aritmetica di Noether dei sistemi lineari di curve algebriche piane.

**Atti della Reale Accademia dei Lincei,** (5), Vol. 27, Fasci. 8-12, (1 Semestre), 1918.

Bompiani, Le trasformazioni puntuali di una varietà che conservano le superficie a curvatura nulla. Burali-Forti, Sulle superficie rigate. Pérès, Quelques propriétés des fonctions de Bessel. Amerio, Pireliometro integrale. Bianchi, Sopra certe forme particolari d'elemento lineare sferico. Cisotti, Forma intrinseca delle equazioni gravitazionali nella relatività generale. Marletta, Di una classe di forme dell' $S_4$ ognuna rappresentabile nelle coppie di un'involuzione dell' $S_3$. Pérès, Quelques propriétés des fonctions de Bessel. Armellini, Sopra L'integrazione approssimata delle equazioni differenziali. Cisotti, Derivazione intrinseca nel calcolo differenziale assoluto. De Cristfaro, Problemi dinamici a due variabili che ammettono un integrale razionale lineare e fratto rispetto alle componenti della velocità. Montesano, Sulla teoria generale delle corrispondenze birazionali dello spazio. Pérès, Quelques propriétés des fonctions de Bessel. Rosaria Giordano, Enti geometrici coordinati a certi covarianti simultanei estensivi. T. Lalesco, Les équations differentielles linéaires d'ordre infini et l'équation de Fredholm. E. Laura, Sopra una classe di nuclei semi-definiti positivi. Montesano, Sulla teoria generale delle corrispondenze birationali delle spazio.

# 雜　錄　彙　報

## 統計論ノ或ル問題

全ク偶然的ト見做サルベキ事象カラ得ラレタ度數 (frequency) ノ數列 $a_1, a_2, a_3, \ldots a_n$ ガ與ヘラレタ場合,ニ 此ノ數列中ノ一ガ前後ノ數ヨリ大ナルコトノ prob. ハ三分ノ一ナリトイフコトガ, 寺田博士ニヨリテ注意セラレ, 其ノ數學的研究ハ吉江, 田谷, 渡邊氏等ニヨリテ遂ゲラレタ (數物 (2,8, 1916 ニ於ケル寺田, 渡邊氏ノ論文參照). 余ハ或ル機會カラ Fechner ノ Collectivmasslehre, p. 369-366 ニ於テ次ノ事柄ガ述ベラレテアルノニ氣付イタ.

曰ク: $a_1-a_2$, $a_2-a_3, \ldots, a_{n-1}-a_n$ ノ符號ノ配布ニ於テ同符號ガ二ツ續ク prob. ハ $\frac{1}{3}$ デアリ, 符號ノ變化ガ起ル prob. ハ $\frac{2}{3}$ デアルト.

之ハ寺田博士ノ述ベラレタノト外形ハ違フテオルガ中身ハ同ジデアル.

次ニ Lipps ノ Die Theorie der Collectivgegenstände, 1902, ノ第三章平均數ノ性質トイフ所ニ, 正數 $a_0, a_1, a_2, \ldots, a_m$ ノ一種ノ平均數トシテ

$$C_n=\sum_{i=0}^{m} p_i\, a_i^{\,n}, \qquad (p_i \geqq 0,\ \Sigma p_i=1)$$

ノ $n$ 乘根ヲトリ, $C_n$ ノ間ノ種々ノ不等式ガ論ゼラレテオル.

此ノ $C_1, C_2, C_3, \ldots C_{2m}$ ヲ特色附ケル性質ハ, 正符號ヲ有スル二次形式ニ關スル Hurwitz Fischer ノ研究ニ由テ, 二次形式

$$\sum_{i,k}^{0,m} C_{i+k}\, x_i\, x_k$$

ガ正ノ定符號ヲ有スル形式ナルコトデアル. コノコトカラシテ Lipps ガ出シテオル多クノ不等式ガ導カレル, (本誌 6, 1914 余ノ論文參照).

${}_p C_q > C_r\, C_s$, $(p+q=r+s,\ p>r,\ s>q)$ ナル不等式ハ本誌 13. 1918 ニ於ケル余ノ論文中ノ積分ニ關スル不等式ヲ有限個ノ和ノ場合ニ直セバ導キ得ラレル. (M. F.)

## 和算ノ一問題

嘉永四年 1851 ノ出版ニカヽル竹內修敬著算法圓理括發ニ其門人大脇銀次郎豐久ノ自問自答ナリトテ次ノ如キ面白キ問題アリ.

所設ノ圓 $O$ ノ弦ヲ $MN$ トス, $MN$ ヲ外公切線トシ且ツ圓 $O$ ニ內切スル任意ノ二圓 $A_1, B_1$ ヲ作ル, 此二圓ト $MN$ トノ切點ヲ夫々 $D_1, E_1$ トス, $D_1, E_1$ ニ於イテ $MN$ ニ切シ且ツ圓 $O$ ニ內切スル二ツノ圓 $A_2, B_2$ ヲ作ル, 此二圓ノ第二ノ外公切線ガ圓 $A_2, B_2$ ニ切スル點ヲ $D_2, E_2$ トス, $D_2, E_2$ ニテ $D_2E_2$ ト切シ且ツ圓 $O$ ニ內切スル圓 $A_3, B_3$ ヲ作ル, 此手續ヲ繰返シテ圓 $A_n, B_n$ ヲ得タリトス, 今圓 $A_r, B_r$ ノ半徑ヲ夫々 $a_r, b_r$ ニテ表ストキハ

$$a_1 b_n = a_n b_1, \qquad \text{即チ} \qquad a_1 : b_1 = a_n = b_n.$$

大脇ノ解ハ示サレズト雖モ嘉永七年 1854 ノ出版ニカヽル算法通書ニハ計算ニヨリテ

$$a_1 : b_1 = a_2 : b_2$$

ヲ求メ之ヲ幾回モ適用シテ $a_1 : b_1 = a_n : b_n$ ヲ得トセリ.

此結果ハ反形法ニヨリテ極メテ簡單ニ證明セラル.

$MN$ ト $D_2E_2$ トノ交點ヲ $P$ トシ $P$ ヲ中心トシ圓 $O$ ト直交スル圓 $K$ ヲ反形圓トスレバ圓 $O$ ト直線 $D_2E_2$ ハソレ自身ニ反轉セラルヽヲ以テ圓 $A_2, B_2$ ハ互ニ他ノ反形ナリ, 故ニ $D_3E_3$ ハ $P$ ヲ通ル, 然ラバ相似三角形ニヨリテ

$$a_1 : b_1 = D_2P : E_2P = a_2 : b_2$$

此定理ハナホ若干ノ擴張ヲユルス，又 $D_1, D_2, D_3, \ldots$ ハ $P$ ヲ中心トスル圓周 $D$ ノ上ニ，$E_1, E_2, E_3, \ldots$ ハ $D$ ト同心ナル圓 $E$ ノ上ニアリ；圓 $D_1, D_2, D_3, \ldots$ ハ他ノ一ツノ圓 $X$ ニ切シ，$E_1, E_2, E_3, \ldots$ モ亦他ノ一ツノ圓 $Y$ ニ切ス，而シテ $X, Y$ ハ互ニ他ノ反形ナリ (K. Y.)

## 奇ナル關係

雜誌 Mathematical questions and solution, new series, Vol. 6, Part 2, August-September 1918, p. 32 ニ次ノ關係ノ記載セラルルヲ見ル，奇ナリトイフベシ.

$$\tan\left(1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+-\ldots\right)=\tan^{-1}\left(\frac{1}{1}-\frac{1}{3}-\frac{1}{5}-\frac{1}{7}-\ldots\right).$$

## 蘭文天文學飜譯

九州帝國大學工科大學教授桑木博士ノ報ゼラルルトコロニヨレバ「長崎市役所ニ藏セル本木仁太夫，吉雄幸作等ノ譯セル天文書ヲ一見致候ガ彼ノ解體新書ノ完成セシ同年ニ一ツノ天文書モ出來致候事ナド知リ申候，官命ニヨリ飜譯トアリ．全編虫喰ヒ辛フジテ頁ヲ繰ルバカリニ候．寫本ニ御座候」ト，原書名，著者名等ノ研究發表セラレンコトヲ希フ．(T. H.)

## 綜合幾何學ノ問題

余ハ本誌第 14 卷第 101 頁ニ於テ同心圓ノ一系ト一ツノ楕圓トヲ取リタルトキ此ノ楕圓ト其ノ一圓トノ四交點ノ重心ハ其ノ圓ノ選定如何ニ拘ハラズ一定ナリトイヘリ．此ノ定理ノ初等的綜合的證明如何．(T. H.)

## 會田安明，小野友五郎ノ贈位

大正 7 年 (1918) 11 月 18 日陸軍特別大演習御臨幸ヲ機トシ，例ノ通リ多クノ功勞アル故人ニ贈位ノ恩典アリ，會田安明及ビ小野友五郎ハ從五位ヲ贈ラル．會田ノ事蹟ニツキテハ本誌第 5 卷第 69 頁ヲ參照セラルベシ．小野友五郎，名ハ廣紳．舊幕府勘定奉行ニシテ發算測量ノ術ニ通ジ，安政 2 年 (1855) 選バレテ長崎海軍傳習所ニ派遣セラレ，算術ヲ蘭人ニ學ブ，蓋シ此事當時本邦內唯一人ナリ．爾後軍艦操練習所教授，同頭取，遣米航海士，浦賀船渠建設者トシテ我ガ海軍創設ニ力ヲ致シ兵制改革財政匡濟ニ與ツテ力アリ．東京數學物理學會ノ前身ナル東京數學會社ノ發起者ノ一人ナリ．安政元年 (1854) 再刻大全塵劫記附錄ヲ著ハス．蓋シ大全塵劫記ハ天保 3 年 (1832) 長谷川寬ノ高弟ノ山本賀前ノ編スル所ナリ．又小野ハ長谷川社ノ齊長トシテ同社ノ重鎭ナルガ，必用算法，量地圖說，算法叢書等ノ著者ナリ．(T. H.)

## 中等學校數學科教員協議會

本誌前號ニ詳報セル同會ハ中等教育研究會（會長ハ嘉納治五郎氏）ノ主唱ニヨリテ開催セラレタルモノナルガ，會期ハ其時記載セルモノト異ナリ．會場ノ都合ニヨリテ大正七年十二月二十日ヨリ二十四日ニ至ル五日間ニ變更セラレタリ．諮問題，協議題，談話題等ニ關シ頗ル有效ノ成績ヲ收メタリ．時恰モ中學校教科要目ノ改正セラルベキトキニ當ル．本會合ニ於テ決定セル諸事項ガ文部省ニヨリテ容レラルヽコトヲ切望ニ堪エズ．會スルモノ三百ニ垂ンタリシガ鬱積セル平時ノ主張ガ代表的ノ經驗家ニヨリテ十分ニ吐露セラレタルハ蓋シ近時斯界ノ壯觀ナリシ．講演者及其演題ハ次ノ如シ．

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|---|---|---|---|
| 十二月 | 二十一日 | 萬國數學者大會回顧其他雜感 | 藤澤博士 |
| 同 | 二十二日 | 對數ノ計算ニ就テ | 寺尾博士 |
| 同 | 二十三日 | 教材ノ選定ニ就テ | 林　博士 |

尙同協議會ニ於テ本邦ニ於ケル常設的ノ中等教育數學教師ノ團結ヲ必要トスルトノ建議出テヽ全會員ハ可決シ，其準備委員トシテ樺，國枝，波木井，森，林ノ五氏ヲ指定シ更ニ全會員ハ發起者トシテ活動スルコトヽナレリ．其準備委員ハ更ニ各種學校ヲ代表セル人々ヲ選擧シテ共ニ開會ノ步ヲ進メンガタメ大正八年一月十八日ヲ期シテ東京女子高等師範學校內ニ會合スル筈ナリ．(T. H)

## 二三雜誌中ノ注目スベキ論說記事

東京物理學校雜誌，大正 7 年 11, 12 月號

| | |
|---|---|
| 原子容調和解析ニ就テ | 理學博士 小倉金之助氏<br>古澤一夫氏 |
| 紙蔦絲ノ簡單ナル方程式 | 理學士 乙部 孝吉氏 |
| 數學小引(第一回) | 理學博士 小倉金之助氏 |
| 特種絕對向心運動之軌道方程式 | 王 應 偉 |

保險雜誌，大正 7 年 10, 11, 12 月號

| | |
|---|---|
| 生命保險技術ノ發展經路 | 木村伊助氏 |
| 一ツノ補整式ニ就テ | 村松 廣吉氏 |
| 英國小口保險戰時死亡率 | 理學士 竹下 清松氏 |

## 諸 學 者 ノ 消 息

瑞西ばーぜる大學教授へつけ E. Hecke 氏ハげつちんげん大學數學科教授トナレリ．

瑞西べるん大學教授ぐらーふ J. H. Graf 氏ハ 66 歳ニテ逝去セリ．

佛蘭西つらーず大學講師らつて S. Lattès 氏ハ 1918 年 7 月 5 日 43 歳ニテ逝去セリ．

伊太利ぴざ大學ノ天體力學及測地學教授タリシぴあつぇつち P. Piazetti 氏ハ 1918 年 4 月 14 日逝去セリ．

希臘雅典大學教授すてふあのす C. Stephanos 氏ハ 60 歳ニテ逝去セリ．

巴里學士院天文學部長うぉるふ C. J. E. Wolf 氏ハ 91 歳ニテ逝去セリ．

獨逸すとらつすぶるく大學教授しもん M. Simon 氏ハ 1918 年 1 月 74 歳ニテ逝去セリ．

芬蘭へるしんぐふおるす大學教授ねおうぬうす E. R. Neovius 氏ハ 67 歳ニテ逝去セリ，最小曲面ニ關スル氏ノ二研究ハ興味アルモノナリ．

北米合衆國はーばーど大學教授まきしむ ぼーしやー Maxime Bôcher 氏ハ 1918 年 9 月 12 日 51 歳ニテ逝去セリ．

英吉利まんちぇすたー大學教授らむ Horace Lamb 氏ハ次年度ニ於ケルおくすふおるど大學ニ於ケル Halley Lecturer トナレリ．

北米合衆國ばーねつと I. A. Barnett 氏ハわしんとん大學講師トナレリ．

獨逸はるれ大學教授かんとる G. Cantor 氏ハ 1918 年 1 月 6 日 73 歳ニテ逝去セリ．

Mengenlehre 建設者かんとる G. Cantor ハ 1845 年ぺてるすぶるくニ生レ父ハこぺんはーげん生レノ商人デアツタ．だるむすたつと，ちゆりつひ，げつちんげん及べるりんニ學ビ 1867 年べるりんニ於テどくとるトナツタ，ソノ三論文ノーハ數學ニ於テハ問題ノ正當ナル Formulierung ガ重要ナノデアツテ是レ其問題ノ解答ヨリモ更ニ重要ナルモノデアルト云フコトヲ論シタ者デアル，1869 年はるれ大學講師トナリ 1872 年助教授トナリ 1879 年教授トナツタ．氏ハ研究的才能ニ秀デテ居ツタ事ハ夙ニわいえるすとらつす及はいねニ依リテ認メラレタ．「はいねハかんとるガはるれニ移ツタ頃彼ヲ評シ銳敏ナ頭腦ト非凡ナル想像力ヲ有スルガ故ニ將來偉大ナル研究ヲナスデアロウト云ツテ居ツタガ果シテソノ言ノ如クデアツタ」トハ彼レノ七十歳祝賀ノ際ニナサレタぐつつまあ教授ノ談デアル．氏ノ研究ハヨク知ラレテ居ル通リ Mengenlehre ニ關スル事ガソノ大部分デアツテ現今函數論ノ發達及代數的及超越的數ノ理論

ニ非常ナ貢獻ヲナシタ事ハ疑モナイ，然モ猶斯ノ如キ研究ヲ發表スルニ際シ數年間躊躇シテ居ツタト云フコトハ彼ガ偉大ナル學者デアツタ事ヲ示シテ居ル.

其他 Crelle 72, Math. Ann. 5 ニ三角級數ニ關スル論文ニ於テ與ヘラレタル實函數 $f(x)$ ハ唯一ノ仕方ニテ三角級數ニ展開シ得ルノミト云フ定理及ソノ擴張ヲ論ジ又 Math. Ann. 4 及 16 ニ於テ $(\alpha, \beta)$ ナル區域ニ於テ各ノ $x$ ノ特別ノ値ニ對シ $\lim\limits_{n=\infty}(a_n \cos nx + b_n \sin nx) = 0$ ナルトキハ $\lim\limits_{n=\infty} a_n = 0$ $\lim\limits_{n=\infty} b_n = 0$ ナルコトノ二三ノ證明ヲナシタ是等ハ凡テ三角級數ヲ論ジテアル教科書ニ載セテアル興味アル定理デアル.

大正 7 年 (1918) 10 月 25 日廣島高等師範學校教諭兼教授津山三郎氏ハ數學及數學教授法研究ノ爲米國ヘ滿二ケ年間留學ヲ命ゼラレタリ，大正 8 年ニ入リテ渡航セラルト聞ク.

大正 7 年 (1918) 10 月 30 日東京高等師範學校名譽教授千本福隆氏逝去セラル，氏ハ明治 12 年 (1879) 東京理科大學卒業同 18 年 (1885) 歐洲ニ留學シ歸朝後東京高師教授ニ任ゼラレ大正 3 年 (1914) 7 月退官名譽教授トナル. 正四位勳三等タリ. 享年六十五歳. 其ノ高師ニ入ルノ前ハ陸軍砲工學校ニ奉職セラレタリト.

大正 7 年 (1918) 12 月 23 日熊本第五高等學校教授米山國藏氏ハ連續的點集合論ト題スル論文ヲ提出シ東北帝國大學理科大學教授會ノ審査ヲ經テ理學博士ノ學位ヲ授ケラレタリ. 同論文ハ本誌第 12 卷及ビ第 13 卷ニ跨リテ出版セラレタルモノナリ.

同年同月同日遞信省技師龜田豐治朗氏ハ母函數ノ理論及其確率論上ノ應用ト題スル論文ヲ提出シ東京帝國大學理科大學教授會ノ審査ヲ經テ理學博士ノ學位ヲ授ケラレタリ. 同論文ハ東京數學物理學會記事第八卷 (1915) ニ掲載セラレタルモノナリト仄聞ス. 龜田氏ハ東京大學ノ出身ニシテ前記米山氏ハ京都大學ノ出身ナルガ，共ニ常ニ本誌ニ有益ナル論文ヲ寄セラルルコト讀者ノ知ル所ナルベシ.

# Note on Laguerre Transformations,

by

TADAHIKO KUBOTA, Sendai.

Mr. R. Lachlan (1) gives the following definition for the circular reciprocation: "Let $S$ denote a fixed circle and $P$, $P'$ be a pair of inverse points with respect to $S$. Then there can be found one circle, which cuts $S$ orthogonally and is coaxal with the system $\{S, P, P'\}$. This circle we shall call the reciprocal with regard to $S$ of the point-pair $P$, $P'$; or simply the reciprocal of the point $P$. The circle $S$ will be called the circle of reciprocation. The reciprocal circle of a point will evidently be a real circle only when the circle of reciprocation is imaginary..... when the locus of a point $P$ is a circle, the reciprocal of the point $P$ will envelope two circles, constituting a pair of inverse circles with respect to the circle of reciprocation. These circles will be called the reciprocal of the circle which is the locus of $P$."

If we observe this transformation more minutely, then it may be looked upon as a contact transformation of oriented circles. In this note I should like to show a close relation between the circular reciprocation and the Laguerre transformation and that a circular reciprocation can always be factored into dilatations and an inversion. The Laguerre transformation is to be defined as equilong contact transformation by which oriented circles are transformed into oriented circles. If we consider, as usual, an oriented circle on the $z$ plane as the minimal projection of a point in space, then the Laguerre transformation corresponds to a motion or a motion composed with a reflexion with regard to a plane in space. The Laguerre transformation is said to be direct or indirect according as the corresponding space transformation is a motion or a motion composed with a reflexion with regard to a plane.

It was proved by Mr. W. Blaschke (2) that every direct Laguerre transformation can be factored into four Laguerre inversions. To a similar transformation (a motion is a special case of it when nothing else

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(1) Lachlan, Modern pure geometry, 1893.
Larmor, Proc. London Math. Soc. 23. 1887.
K. Ogura, Proc. Tôkyô Math. Phys. Soc. 1909.

(2) W. Blaschke, Monatshefte für Math. u. Phys. 1910, Grünwald, Monatshefte für Math. u. Phys. 1906.

is stated) or to a similar transformation composed with a reflexion in space there corresponds a contact transformation of oriented circles on the $z$ plane by which the length of the common tangent of any two oriented circles will be multiplied by a constant. Such a transformation is nothing but a Laguerre transformation composed with a similar (similarly situated figures) transformation. We shall first show the following

Theorem I. *Every contact transformation on a plane, by which oriented circles are transformed into oriented circles, can be factored into Laguerre transformations, similar transformations and circular reciprocations.*

If we let correspond a point in space into an oriented circle in the $z$ plane by the minimal projection, then every contact transformation of oriented circles in the $z$ plane may be obtained by the minimal projection of a certain conformal point transformation in space, which has been proved by Lie ([1]). In proving theorem I it will be convenient to define the circular reciprocation in the following way. Consider the sphere whose equation in rectangular coordinates is

$$x^2+y^2+z^2+R^2=0. \tag{1}$$

Project the plane figure in the $z$ plane stereographically from the point $(0, 0, Ri)$ upon the sphere (1). Then project the figure thus obtained upon the $z$ plane by the minimal projection. In this manner every point on the $z$ plane will be transformed into an oriented circle on the same plane. This transformation must necessarily, by Lie's theorem, be a contact transformation of oriented circles. We shall prove that this transformation is nothing but the circular reciprocation with respect to the fundamental circle

$$x^2+y^2+R^2=0, \tag{2}$$

expressed only in a more precise form.

If we take two inverse points $P$ and $P'$ with respect to the circle (2), then the stereographic projections $p$, $p'$ of the points $P$ and $P'$ on the sphere (1) are symmetrical with respect to the $z$ plane. These points $p, p'$ give by the minimal projection one and the same circle oriented in opposite senses. If we take an oriented circle $C$ on the $z$ plane, there corresponds on the sphere (1) by the stereographic projection an oriented circle $K$. If we project this circle by the minimal projection on the $z$ plane then an oriented circle will be obtained which is the envelope of the minimal projections of points on $K$.

([1]) Lie-Scheffers, Geometrie der Berührungstransformationen, p. 426.

More minutely we may proceed as follows. If the point $Q$ representing the oriented circle $C$ on the $z$ plane be inverted into the point $q$ with respect to the sphere

$$x^2+y^2+z^2-2\,Ri\,z+R^2=0, \tag{3}$$

then the minimal projection of the point $q$ gives an oriented circle which is an envelope of the minimal projections of points of the oriented circle $K$. Let $(x, y)$ be a point on the $z$ plane, then the stereographic projection is

$$\left(\frac{-2\,x\,R^2}{x^2+y^2-R^2},\ \frac{-2\,y\,R^2}{x^2+y^2-R^2},\ iR\frac{x^2+y^2+R^2}{x^2+y^2-R^2}\right).$$

Its minimal projection is the oriented circle

$$\xi^2+\eta^2+\frac{4\,R^2\,(x\,\xi+y\,\eta)}{x^2+y^2-R^2}-R^2=0,$$

whose radius is equal to

$$R\frac{x^2+y^2+R^2}{x^2+y^2-R^2}.$$

From this fact follows the equivalence of our transformation with the circular reciprocation. Next consider the oriented circle

$$(\xi-x)^2+(\eta-y)^2=r^2, \tag{4}$$

whose radius is equal to $r$; this oriented circle corresponds evidently to the point $(x, y, i\,r)$, whose inverse point with respect to the sphere (3) is readily shown to be

$$\left(\frac{-2\,R^2\,x}{x^2+y^2-(R-r)^2},\ \frac{-2\,R^2\,y}{x^2+y^2-(R-r)^2},\ iR\,\frac{x^2+y^2+R^2-r^2}{x^2+y^2-(R-r)^2}\right).$$

The corresponding oriented circle is

$$\left(\xi+\frac{2\,R^2\,x}{x^2+y^2-(R-r)^2}\right)^2+\left(\eta+\frac{2\,R^2\,y}{x^2+y^2-(R-r)^2}\right)^2$$
$$=\left(R\frac{x^2+y^2+R^2-r^2}{x^2+y^2-(R-r)^2}\right)^2, \tag{5}$$

and has the radius equal to

$$R\,\frac{x^2+y^2+R^2-r^2}{x^2+y^2-(R-r)^2}.$$

The fundamental property of the circular reciprocation that when the point $P$ describes a circle the reciprocal of the point $P$ envelopes two circles etc. is thus opportunely proved from an entirely different point of view. After this preparation we now go on to prove theorem I; since any contact transformation of oriented circles can be obtained by the minimal projection of a certain conformal point transformation in space, which can be factored into similar transformations ([1]) and inversions, any contact transformation of oriented circles can be factored into similar transformations (of similar situations), Laguerre transformations and circular reciprocations. Hence theorem I is proved.

Theorem II. *Every circular reciprocation can be factored into dilatations and an inversion.*

If we represent the translation along the $z$ axis

$$z' = z + Ri$$

by $T$ and the inversion with respect to

$$x^2 + y^2 + z^2 + 2R^2 = 0$$

by $I'$ then the inversion with respect to the sphere (3) can be put in the form

$$T^{-1} I' T,$$

consequently any circular reciprocation can be factored into 2 dilatations and an inversion. Hence theorem II is proved. To verify the result we take the oriented circle

$$(\xi - x)^2 + (\eta - y)^2 = r^2$$

whose radius is $r$. Apply the dilatation of distance $-R$ then the oriented circle is transformed into the oriented circle of radius $r - R$:

$$(\xi - x)^2 + (\eta - y)^2 = (r - R)^2.$$

Next apply the inversion with respect to the circle

$$\xi^2 + \eta^2 + 2R^2 = 0$$

then we get the oriented circle

$$\left(\xi + \frac{2R^2 x}{x^2 + y^2 - (r-R)^2}\right)^2 + \left(\eta + \frac{2R^2 y}{x^2 + y^2 - (r-R)^2}\right)^2 = \left(\frac{2R^2(R-r)}{x^2 + y^2 - (r-R)^2}\right)^2$$

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([1]) Motions are included in similar transformations.

whose radius is

$$\frac{2R^2(R-r)}{x^2+y^2-(r-R)^2}.$$

Apply again the dilatation of distance $R$; then we get the oriented circle

$$\left(\xi+\frac{2R^2x}{x^2+y^2-(r-R)^2}\right)^2+\left(\eta+\frac{2R^2y}{x^2+y^2-(r-R)^2}\right)^2$$
$$=\left(R\frac{x^2+y^2+R^2-r^2}{x^2+y^2-(r-R)^2}\right)^2$$

whose radius is equal to $R\dfrac{x^2+y^2+R^2-r^2}{x^2+y^2-(r-R)^2}$. This is the same as (5). It will be almost unnecessary to remark here that the method admits of extensions to the higher spaces.

Combining theorems I and II we get Lie and Scheffers'(1) theorem that every contact transformation of oriented circles can be factored into Laguerre transformations, similar transformations and inversions, dilatations being included in Laguerre transformations.

(1) G. Scheffers, Synthetische Bestimmung aller Berührungstransformationen des Kreises in der Ebene, Leipziger Berichte, 1899.

# On Steiner's Problem of Closure,

by

KITIZI YANAGIHARA, Sendai.

In elementary geometry we find the following proposition.

(1) Let $F, I$; $D, G$; $E, H$ be six points taken respectively on the sides $BC$, $CA$, $AB$ of the triangle $ABC$, so that

$$DE \parallel BC,\ EF \parallel CA,\ FG \parallel AB$$

$$GH \parallel BC,\ HI \parallel CA,$$

then $ID$ is also parallel to $AB$.

This suggests the following theorem.

(2) If six points $D$, $E$, $F$, $G$, $H$, $I$ be so taken on a plane cubic curve $\Gamma$ having three asymptotes $u$, $v$ and $w$, that

$$DE \parallel u,\ EF \parallel v,\ FG \parallel w,$$

$$GH \parallel u,\ HI \parallel v,$$

then $ID$ is also parallel to $w$.

More generally we have:

(3) Let a cubic curve $\Gamma$ be cut by a straight line $u$, in three points $I$, $II$, $III$, and let six points $A_1$, $A_2$, $A_3$, $\cdots\cdots$, $A_6$ be so taken on $\Gamma$, that

$$A_1,\ A_2,\ III;\quad A_2,\ A_3,\ I;\quad A_3,\ A_4,\ II;$$

$$A_4,\ A_5,\ III;\quad A_5,\ A_6,\ I,$$

are all collinear, then $A_6$, $A_1$ and $II$ will also be collinear.

This can be put in a more general form as follows.

(4) Let $I$, $II$, $III$ be three points arbitrarily taken on $\Gamma$, and let $A_1$, $A_2$, $A_3$, $\cdots\cdots$, $A_6$ be six points on $\Gamma$, such that

$$A_1,\ A_2,\ III;\quad A_2,\ A_3,\ I;\quad A_3,\ A_4,\ II;$$

$$A_4,\ A_5,\ III;\quad A_5,\ A_6,\ I,$$

are all collinear, then $A_6$, $A_1$ and $II$ are also collinear.

It can be easily seen that the last is another form of enunciation of the following well known theorem.

(5) If each two pairs of opposite sides of hexagon, inscribed in a

cubic $\Gamma$, have points on $\Gamma$ in common respectively, then the remaining pair of opposite sides will also have a point on $\Gamma$ in common.

Hence the validity of (1), (2), (3) and (4) is obvious.

Next, we will try to extend (4) in the following direction.

(6) Let two straight lines $u$, $v$ cut $\Gamma$ in $I$, $II$, $III$; $I'$, $II'$, $III'$ respectively, and let $A_1$, $A_2$, $A_3$, $B_1$, $B_2$, $B_3$ be taken on $\Gamma$ so that

$$A_1,\ A_2,\ I;\quad A_2,\ A_3,\ II';\quad A_3,\ B_1,\ III;$$

$$B_1,\ B_2,\ I';\quad B_2,\ B_3,\ II$$

are collinear, then $B_3$, $A_1$, $III'$ are also collinear.

It can be easily verified that the last can be stated in a modified form as follows.

(6′) If two triangles $A_1 A_2 A_3$ and $B_1 B_2 B_3$ be inscribed in a cubic $\Gamma$, and

$$A_1 A_2,\ B_1 B_2;\quad A_2 A_3,\ B_2 B_3$$

respectively meet in a point on $\Gamma$, then $A_3 B_1$, $B_3 A_1$ meet in a point on $\Gamma$.

This is also another form of enunciation of theorem (5).

Next we will proceed to the following theorem.

(7) Let two quadrilaterals $A_1 A_2 A_3 A_4$, $B_1 B_2 B_3 B_4$ be inscribed in a cubic $\Gamma$, in such a way that

$$A_1 A_2,\ B_1 B_2;\quad A_2 A_3,\ B_2 B_3;\quad A_3 A_4,\ B_3 B_4$$

respectively meet in a point on $\Gamma$, then $A_4 A_1$ and $B_4 B_1$ also meet in a point on $\Gamma$.

For since $A_1 A_2$, $B_1 B_2$; $A_2 A_3$, $B_2 B_3$ respectively meet in a point on $\Gamma$, by (6′) $A_3 B_1$ and $B_3 A_1$ meet in a point on $\Gamma$. Then since

$$A_1 B_3,\ B_1 A_3;\quad B_3 A_4,\ A_3 B_4$$

respectively meet in a point on $\Gamma$, by (6′) $A_1 A_4$, $B_1 B_4$ also meet in a point on $\Gamma$.

Then by mathematical induction, we arrive at the following theorem.

(8) Let two polygons of $n$ sides $A_1 A_2 A_3 \cdots\cdots A_n$ and $B_1 B_2 B_3 \cdots\cdots B_n$ be inscribed in a cubic $\Gamma$, in such a way that

$$A_r A_{r+1},\ B_r B_{r+1}\quad (r=1, 2, 3, \cdots\cdots, n-1)$$

respectively meet in a point on $\Gamma$, then $A_n A_1$, $B_n B_1$ or $A_n B_1$, $A_1 B_n$ meet in a point on $\Gamma$, according as $n$ is even or odd.

If $n$ be odd, $A_1 A_2 A_3 \cdots\cdots A_n B_1 B_2 B_3 \cdots\cdots B_n$ forms a closed poly-

gon of $2n$ sides inscribed in $\Gamma$, and therefore (8) can be regarded as an extension of (5) ([1]).

If $n$ be even, it is nothing but an extension of Steiner's problem of closure ([2]).

(9) Next, we will try to extend (6) as follows: Let $k$ be an even integer which is not divisible by 3, and let the points of intersection of the straight line $L_i$ with a cubic $\Gamma$ be denoted by

$$A_i,\ B_i,\ C_i, \qquad i=1, 2, 3, \cdots\cdots, k.$$

On $\Gamma$, take $3k+1$ points

$$P_1,\ P_2,\ \cdots\cdots\ P_k,\ Q_1,\ Q_2,\ \cdots\cdots\ Q_k,\ R_1,\ R_2,\ \cdots\cdots\ R_k,\ R$$

in such a way that for $k=3n+1$,

$$\begin{array}{llll}
P_1, & A_1, & P_2; & P_2, \quad B_2, \quad P_3; \\
P_3, & C_3, & P_4; & P_4, \quad A_4, \quad P_5; \\
P_5, & B_5, & P_6; & P_6, \quad C_6, \quad P_7; \\
\cdots & \cdots & \cdots & \cdots\cdots\cdots\cdots \\
P_{3n}, & C_{3n}, & P_{3n+1}; & P_{3n+1}, \quad A_{3n+1}, \quad Q_1; \\
Q_1, & B_1, & Q_2; & Q_2, \quad C_2, \quad Q_3; \\
Q_3, & A_3, & Q_4; & Q_4, \quad B_4, \quad Q_5: \\
\cdots & \cdots & \cdots & \cdots\cdots\cdots\cdots \\
Q_{3n}, & A_{3n}, & Q_{3n+1}; & Q_{3n+1}, \quad B_{3n+1}, \quad R_1; \\
R_1, & C_1, & R_2; & R_2, \quad A_2, \quad R_3; \\
R_3, & B_3, & R_4; & R_4, \quad C_4, \quad R_5; \\
\cdots & \cdots & \cdots & \cdots\cdots\cdots\cdots \\
R_{3n}, & B_{3n}, & R_{3n+1}; & R_{3n+1}, \quad C_{3n+1}, \quad R
\end{array}$$

are all collinear, and for $k=3m+2$,

$$\begin{array}{llll}
P_1, & A_1, & P_2; & P_2, \quad B_2, \quad P_3; \\
P_3, & C_3, & P_4; & P_4, \quad A_4, \quad P_5; \\
P_5, & B_5, & P_6; & P_6, \quad C_6, \quad P_7; \\
\cdots & \cdots & \cdots & \cdots\cdots\cdots\cdots \\
P_{3m+1}, & A_{3m+1}, & P_{3m+2}; & P_{3m+2}, \quad B_{3m+2}, \quad Q_1; \\
Q_1, & B_1, & Q_2; & Q_2, \quad C_2, \quad Q_3;
\end{array}$$

([1]) ([2]) Schröter's Theorie der ebenen Kurven dritter Ordnung, § 31.

| | |
|---|---|
| $Q_3,\quad A_3,\quad Q_3;$ | $Q_4,\quad B_4,\quad Q_6;$ |
| ..... | ..... |
| $Q_{3m+1}, B_{3m+1}, Q_{3m+2};$ | $Q_{3m+2}, C_{3m+2}, R_1;$ |
| $R_1,\quad C_1,\quad R_2;$ | $R_2,\quad A_2,\quad R_3;$ |
| $R_3,\quad B_3,\quad R_4;$ | $R_4,\quad C_4,\quad R_5;$ |
| ..... | ..... |
| $R_{3m+1}, C_{3m+1}, R_{3m+2};$ | $R_{3m+2}, A_{3m+2}, R$ |

are all collinear, then $R$ coincides with $P_1$.

This can at once be recognized by supposing these letters as representing, at the same time, the corresponding values of the parameter when the coordinates of these points are expressed by elliptic functions, and subtracting the sum of the values in the right column from that of the values in the left column.

We have here $4k$ lines, $3k$ of which are those mentioned above in two columns, and $k$ of which are $L_1, L_2, \cdots\cdots, L_k$. If $k=2$ or 4, by actual examination, we find that these $4k$ lines can be separated into two classes, each forming a polygon inscribed in $\Gamma$, whose sides two by two meet in succession in points on $\Gamma$, so that these special cases are the same as the extension of Steiner's problem of closure in (8). This leads us to imagine that (9) can in general be reduced to (8) by suitable separation of these $4k$ lines into two classes. But as to how this separation can be effected, I have not yet reached to any definite rule.

# On Curves with Monotonous Curvature,

by

TSURUICHI HAYASHI, Sendai.

When the directions and curvatures at the end points of a curve with monotonously increasing (or decreasing) curvature are given, some conditions about the relative position of the two end points must exist. Profs. A. Kneser (1) and W. Vogt (2) have obtained a necessary condition respectively, and Mr. S. Kakeya (3) has proved by applying his elegant theorem on certain integral equations, that the two conditions combined are sufficient for the existence of an analytic curve joining the two points given according to the two conditions. The question is thus already completely solved. To increase the number of papers on this question is now quite unnecessary and is even superfluous, when no more knowledge is added and when only a looser condition is dealt with. From this point of view this paper is wortbless at all. But I believe there are occasions pressed to show the existence of the restrictions about the relative position of the two end points in calculus courses for beginners. This is the reason why I will publish this paper. Moreover I will add that this paper contains only an immediate consequence of the following theorem due to Mr. J. F. Steffensen, the enunciation and proof of which are reproduced here after a very slight modification, since his work (4) is not easily accessible for us.

Besides the said problem I will treat a problem on the area included between a curve and its involute.

Theorem. *If the functions $f(\theta)$ and $\varphi(\theta)$ are integrable in their definition-interval $a<\theta<b$, and the function $f(\theta)$ never increases, and the function $\varphi(\theta)$ satisfies the condition*

$$0 \leqq \varphi(\theta) \leqq r, \quad r \text{ const.},$$

*then the inequalities*

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(1) Festschrift H. Weber, 1912, S. 170.

(2) Crelle Journal, Vol 144, 1914, S. 239.

(3) This Journal, Vol. 6, 1915, p. 130.

(4) *On certain inequalities between mean values, and their applications to actuarial problems*, Skandinavisk Aktuarietidskrift, 1918, pp. 82–97.

$$r\int_{b-\lambda}^{b} f(\theta)\,d\theta \leqq \int_{a}^{b} f(\theta)\,\varphi(\theta)\,d\theta \leqq r\int_{a}^{a+\lambda} f(\theta)\,d\theta$$

*exist, where*

$$\lambda = \frac{1}{r}\int_{a}^{b} \varphi(\theta)\,d\theta.$$

For,

$$\begin{aligned}
&\int_{a}^{a+\lambda} f(\theta)\,d\theta - \int_{a}^{b} f(\theta)\,\frac{\varphi(\theta)}{r}\,d\theta \\
&\quad = \int_{a}^{a+\lambda}\left\{1-\frac{\varphi(\theta)}{r}\right\} f(\theta)\,d\theta - \int_{a+\lambda}^{b} f(\theta)\,\frac{\varphi(\theta)}{r}\,d\theta \\
&\quad = f(\xi)\int_{a}^{a+\lambda}\left\{1-\frac{\varphi(\theta)}{r}\right\} d\theta - \int_{a+\lambda}^{b} f(\theta)\,\frac{\varphi(\theta)}{r}\,d\theta \\
&\quad = f(\xi)\left\{\lambda - \int_{a}^{a+\lambda}\frac{\varphi(\theta)}{r}\,d\theta\right\} - \int_{a+\lambda}^{b} f(\theta)\,\frac{\varphi(\theta)}{r}\,d\theta \\
&\quad = f(\xi)\left\{\int_{a}^{b}\frac{\varphi(\theta)}{r}\,d\theta - \int_{a}^{a+\lambda}\frac{\varphi(\theta)}{r}\,d\theta\right\} - \int_{a+\lambda}^{b} f(\theta)\,\frac{\varphi(\theta)}{r}\,d\theta \\
&\quad = f(\xi)\int_{a+\lambda}^{b}\frac{\varphi(\theta)}{r}\,d\theta - \int_{a+\lambda}^{b} f(\theta)\,\frac{\varphi(\theta)}{r}\,d\theta \\
&\quad = \int_{a+\lambda}^{b}\left\{f(\xi)-f(\theta)\right\}\frac{\varphi(\theta)}{r}\,d\theta \\
&\quad \geqq 0,
\end{aligned}$$

where $\xi$ lies between $a$ and $a+\lambda$.

The second part of the required inequalities is thus proved. For the first part a similar reasoning is sufficient.

Now draw a curve with monotonously increasing curvature between two points $A$ and $B$, the radii of curvature at these points being $\rho_a$ and $\rho_b$ respectively, and take the point $A$ as origin and the tangent to the curve at the point $A$ as the $x$-axis, and denote the angle which the tangent at a point $M$ on the curve between $A$ and $B$ makes with the $x$-axis by $\theta$, so that $\theta = a = 0$ at $A$ and $\theta = b \leqq \frac{\pi}{2}$ at $B$. Then by applying the radius of curvature and $\cos\theta$ or $\sin\theta$ for the functions $f(\theta)$ and $\varphi(\theta)$ respectively in the above inequalities, we get the following:

$$\rho_b \sin b \leqq X \leqq \rho_a \sin b,$$

$$\rho_b(1-\cos b) \leqq Y \leqq \rho_a(1-\cos b),$$

where $X$ and $Y$ stand for the rectangular coordinates of the end point $B$.

Taking advantage of this opportunity I will record here the bounds between which lies the area enclosed by a curve with monotonous curvature, its involute and the radii of curvature at its two end points. This area is expressed by

$$\frac{1}{2}\int_0^b \rho^2\, d\theta$$

according to the above adopted axes of coordinates and the notations. Taking $\rho$ for the both functions $f(\theta)$ and $\varphi(\theta)$ in the general formula, we have

$$\rho_b \leqq \rho \leqq \rho_a$$

and hence we have

$$\frac{1}{2}\rho_a\int_{b-\lambda}^b \rho\, d\theta \leqq \frac{1}{2}\int_a^b \rho^2\, d\theta \leqq \frac{1}{2}\rho_a\int_a^{a+\lambda} \rho\, d\theta,$$

where

$$\lambda = \frac{1}{\rho_a}\int_a^b \rho\, d\theta = \frac{l}{\rho_a},$$

$l$ being the length of the given curve between the end points $A$ and $B$.

Therefore if we denote the length along the given curve from $A$ to the point $M$ on the curve by $[\theta]$, so that

$$[0]=0, \quad [b]=l,$$

we arrive at

$$\frac{\rho_a}{2}\left\{ l-\left[b-\frac{l}{\rho_a}\right]\right\} \leqq \text{the area} \leqq \frac{\rho_a}{2}\left[\frac{l}{\rho_a}\right].$$

This formula will give us sharper bounds than the more readilly attainable formula

$$\frac{1}{2}\rho_b\, l \leqq \text{the area} \leqq \frac{1}{2}\rho_a\, l.$$

The area here treated is equal to that of the associate curve of the given one and its two radii vectores, which was defined by me as the

curve derived from the given one by lengthening or shortening the perpendicular let fall from the origin of coordinates on the tangent of the given curve so as to make it equal to the radius of curvature $\rho$ at the point of contact, in the Science Reports of the Tôhoku Imp. Univ., Vol. 5, 1916, p. 307, and in the Tôhoku Math. Journal, Vol. 12, 1917, p. 218.

July 1918.

# Note on Determinants whose Matrix is that of an Orthogonant increased or diminished by Matrix Unity,

by

THOMAS MUIR, Rondebosch, near Cape Town, S.A.

**1.** Cayley's orthogonant, it will be remembered, is

$$\begin{vmatrix} \frac{2L_{11}}{\Delta}-1 & \frac{2L_{12}}{\Delta} & \cdots\cdots \\ \frac{2L_{21}}{\Delta} & \frac{2L_{22}}{\Delta}-1 & \cdots\cdots \\ \cdots\cdots & \cdots\cdots & \cdots\cdots \end{vmatrix}_n$$

where $|L_{11}L_{22}\cdots L_{nn}|$ is the adjugate of any unit-axial skew determinant $|l_{1n}|$ or $\Delta$. Now, if unity be added to each of its diagonal elements, the resulting determinant is seen to have elements proportional to the elements of

$$|L_{11}L_{22}\cdots L_{nn}|.$$

Consequently, by an old theorem of Cauchy's (1812), the adjugate of the said resulting determinant has its elements proportional to the elements of

$$|l_{11}l_{22}\cdots l_{nn}|,$$

and therefore, like this last determinant, must be skew. We thus learn that *if the diagonal elements of a Cayley's orthogonant be increased by unity the adjugate of the resulting determinant is skew.*

**2.** Whether with the help of this as a suggestion or not, the passage from a Cayley's orthogonant to orthogonants in general has been made, the first draft of the wider theorem being due to Siacci (1872) ([1]). In its latest and fullest form it stands thus: *The determinant,* $\Delta$ *say, formed by adding* 1 *to each element of the main diagonal of an orthogonant has its*

([1]) Teorema sui determinanti ed alcune sue applicazioni, *Atti....Accad....di Torino* VII, pp. 772-783.

Intorno ad alcune trasformazioni di determinanti, *Annali di Mat.* (2) V, pp. 296-304.

*adjugate skew or axisymmetric according as the orthogonant is positive or negative, each diagonal element of the adjugate in the former case being* $\frac{1}{2}\Delta$.

For a number of years this has been fully established: it is still worth while, however by means of an example to draw attention to an exceedingly simple way of proving the first part of it, the essential fact used being that the multiplication of any primary minor of the determinant in question by the associated orthogonant produces the conjugate minor. Taking the case of the fourth order, the orthogonant being $|a_1 b_2 c_3 d_4|$, we write the cofactor of $a_2$ in

$$\begin{vmatrix} a_1-1 & a_2 & a_3 & a_4 \\ b_1 & b_2-1 & b_3 & b_4 \\ c_1 & c_2 & c_3-1 & c_4 \\ d_1 & d_2 & d_3 & d_4-1 \end{vmatrix}$$

in the form

$$\begin{vmatrix} . & 1 & . & . \\ b_1 & b_2-1 & b_3 & b_4 \\ c_1 & c_2 & c_3-1 & c_4 \\ d_1 & d_2 & d_3 & d_4-1 \end{vmatrix} \text{ and multiply it by } \begin{vmatrix} a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \\ c_1 & c_2 & c_3 & c_4 \\ d_1 & d_2 & d_3 & d_4 \end{vmatrix}$$

with the result

$$(\text{cof } a_2).\ |a_1 b_2 c_3 d_4| = \begin{vmatrix} a_2 & b_2 & c_2 & d_2 \\ -a_2 & 1-b_2 & -c_2 & -d_2 \\ -a_3 & -b_3 & 1-c_3 & d_3 \\ -a_4 & -b_4 & -c_4 & 1-d_4 \end{vmatrix}$$

$$= (-1)^3 \begin{vmatrix} a_2 & b_2 & c_2 & d_2 \\ . & -1 & . & . \\ a_3 & b_3 & c_3-1 & d_3 \\ a_4 & b_4 & c_4 & d_4-1 \end{vmatrix} = (-1)^4 \begin{vmatrix} a_2 & c_2 & d_2 \\ a_3 & c_3-1 & d_3 \\ a_4 & c_4 & d_4-1 \end{vmatrix}$$

$$= (-1)^5.\ (\text{cof } b_1);$$

so that cof $a_2$ and cof $b_1$ differ only in sign or are identical according as $|a_1 b_2 c_3 d_4|$ equals $+1$ or $-1$.

In addition to this it is well to know the actual expanded form which the elements (cof. $a_2$, cof. $b_1$, $\cdots\cdots$) of the adjugate take, as there is thence derived additional conviction and even the elements of another proof of the skewness and symmetry referred to. Freed from binomial elements

$$\text{cof } a_2 = -|b_1 c_3 d_4| + |b_1 c_3| + |b_1 d_4| - b_1,$$

and

$$\text{cof } b_1 = -a_2 + |a_2 d_4| + |a_2 c_3| - |a_2 c_3 d_4|,$$

the second expression being written in reverse order from the first so as to have each term of it under the term of equal magnitude in the first. As for the diagonal elements in the case of the skew adjugate, they are all equal to

$$1-(a_1+b_2+c_3+d_4)+|a_1 b_2|+|a_1 c_3|+|a_1 d_4|,$$

the double of this, namely

$$1-\Sigma a_1+\Sigma |a_1 b_2|-\Sigma |a_1 b_2 c_3|+|a_1 b_2 c_3 d_4|$$

being the expansion of $\Delta$.

**3.** It will be observed that the theorem only refers to the diagonal elements in the case where the adjugate is skew. In the other case the diagonal elements are not all equal, but what is more important to note is the fact that the adjugate is then equal to 0, since $\Delta$ itself is then 0; for example, when $|a_1 b_2 c_3|$ is a negative orthogonant the vanishing adjugate is

$$\begin{vmatrix} -a_1+b_2+c_3+1 & -a_2-b_1 & -a_3-c_1 \\ -b_1-a_2 & a_1-b_2+c_3+1 & -b_3-c_2 \\ -c_1-a_3 & -c_2-b_3 & a_1+b_2-c_3+1 \end{vmatrix}.$$

**4.** It is next to be noted that the group of results forming this theorem is not unique, there being a companion theorem which may be formulated in a similar fashion, namely, *the determinant, $\Delta_n$ say, formed by subtracting* 1 *from each element of the main diagonal of an orthogonant $\Omega_n$ has its adjugate skew or axisymmetric according as $(-1)^n \Omega_n$ is positive or negative, each diagonal element of the former case being $-\frac{1}{2}\Delta_n$, and the adjugate in the latter case being equal to* 0.

As for demonstration it suffices to note that all modes of verification and proof used in connection with the former theorem apply equally here.

**5.** A further link of connection between the two determinants,—the one got from an orthogonant by annexing positive units to the diagonal elements and the other by annexing negative units,—is found in the fact that *their product is a zero-axial skew determinant, and consequently is a square when the order of the determinants is even.* For example, we have

$$\begin{vmatrix} a_1+1 & a_2 & a_3 & a_4 \\ b_1 & b_2+1 & b_3 & b_4 \\ c_1 & c_2 & c_3+1 & c_4 \\ d_1 & d_2 & d_3 & d_4+1 \end{vmatrix} \cdot \begin{vmatrix} a_1-1 & a_2 & a_3 & a_4 \\ b_1 & b_2-1 & b_3 & b_4 \\ c_1 & c_2 & c_3-1 & c_4 \\ d_1 & d_2 & d_3 & d_4-1 \end{vmatrix}$$

$$= \begin{vmatrix} a_2-b_1 & a_3-c_1 & a_4-d_1 \\ & b_3-c_2 & b_4-d_2 \\ & & c_4-d_3 \end{vmatrix}^2.$$

That the product vanishes when the order is odd is in accordance with the fact that *every odd-ordered orthogonant* $|a_{1n}|$, *positive or negative, has* $|a_{1n}|$ *for a latent root.*

**6.** An interesting alternative for the Pfaffian is

$$|a_2 c_4| - |a_3 b_4| + |a_3 d_2| + |b_1 d_3| + |b_4 c_1| - |c_1 d_2|,$$

this being composed of *the two-line minors that do not contain a diagonal element.* As it is separable into three minors and their complementaries, namely, into

$$|a_2 c_4| - |a_3 b_4| + |b_1 c_4| \text{ and } |b_1 d_3| - |c_1 d_2| + |a_2 d_3|$$

it follows that when $|a_1 b_2 c_3 d_4|$ is a positive orthogonant the Pfaffian is equal to

$$2\{|a_2 c_4| - |a_3 b_4| + |b_1 c_4|\},$$

and when $|a_1 b_2 c_3 d_4|$ is a negative orthogonant the Pfaffian vanishes, this last being in agreement with the fact that *every even-ordered negative orthogonant* $|a_{1n}|$ *has both* $|a_{1n}|$ *and* $|a_{1n}|$ *for latent roots.*

**7.** The product considered in the foregoing paragraph is not only expressible as a square but as *the difference of two squares.* For the expansion of

$$\begin{vmatrix} a_1-x & a_2 & a_3 & a_4 \\ b_1 & b_2-x & b_3 & b_4 \\ c_1 & c_2 & c_3-x & c_4 \\ d_1 & d_2 & d_3 & d_4-x \end{vmatrix}$$

being

$$x^4-x^3\,\Sigma\, a_1+x^2\,\Sigma\,|\,a_1\,b_2\,|-x\,\Sigma\,|\,a_1\,b_2\,c_3\,|+|\,a_1\,b_2\,c_3\,d_4\,|,$$

it is evident that when $x^2=1$ the determinant is equal to

$$\{1+\Sigma\,|\,a_1\,b_2\,|+|\,a_1\,b_2\,c_3\,d_4\,|\}-x.\{\Sigma\,a_1+\Sigma\,|\,a_1\,b_2\,c_3\,|\},$$

and that therefore the said product of § 6 is equal to

$$\{1+\Sigma\,|\,a_1\,b_2\,|+|\,a_1\,b_2\,c_3\,d_4\,|\}^2-\{\Sigma\,a_1+\Sigma\,|a_1\,b_2\,c_3\,|\}^2.$$

**8.** The equatement of the two expressions thus obtained for our product gives the interesting equality

$$\begin{vmatrix} a_2-b_1 & a_3-c_1 & a_4-d_1 \\ & b_3-c_2 & b_4-d_2 \\ & & c_4-d_3 \end{vmatrix}^2+\{\Sigma\,a_1+\Sigma\,|\,a_1\,b_2\,c_3\,|\}^2 \\ =\{1+\Sigma\,|\,a_1\,b_2\,|+|\,a_1\,b_2\,c_3\,d_4\,|\}^2,$$

the next case of which clearly is

$$\begin{vmatrix} a_2-b_1 & a_3-c_1 & \cdots\cdots & a_6-f_1 \\ & b_3-c_2 & \cdots\cdots & b_6-f_2 \\ & & \cdots\cdots\cdots\cdots & \\ & & & e_6-f_5 \end{vmatrix}^2+\{\Sigma\,a_1+\Sigma\,|\,a_1\,b_2\,c_3\,|+\Sigma\,|\,a_1\,b_2\,c_3\,d_4\,e_5\,|\}^2$$

$$=\{1+\Sigma\,|\,a_1\,b_2\,|+\Sigma\,|\,a_1\,b_2\,c_3\,d_4\,|+|\,a_1\,b_2\,c_3\,d_4\,e_5\,f_6\,|\}^2.$$

It is important, however, to note that the validity of this for negative orthogonants is of no value, because both members then vanish: and that for the other class of orthogonants some simplification is possible. In fact each of the three expressions that are squared then contains the factor 2, and this being removed the first example becomes

$$\{|\,a_2\,c_4\,|-|\,a_3\,b_4\,|-|\,b_1\,c_4\,|\}^2+(a_1+b_3+c_3+d_4)^2 \\ =\{1+|\,a_1\,b_2\,|+|\,a_1\,c_3\,|+|\,a_1\,d_4\,|\}^2.$$

and the others as readily follow with the help of the general theorem used above at the beginning of § 6. (*See Transac. R. Soc. of Edinburgh* XLV (1906) pp. 311-321).

Rondebosch, S.A., 21st May 1918.

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# On the Sign and Magnitude of the Coefficients in the Fourier Series, the Sine Series and the Cosine Series,

by

KINNOSUKE OGURA, Osaka.

## On the Fourier series.

**1.** Prof. Fejér [1] proved the following theorem:

Let $y=F(x)\,(0<x<2\pi)$ be a single analytic are joining the two points $[0,\ F(+0)]$ and $[2\pi,\ F(2\pi-0)]$, and let

$$\frac{b_0}{2}+\sum_{n=1}^{\infty}(a_n\sin nx+b_n\cos nx)$$

be the Fourier series corresponding to $F(x)$. If $F(+0)\neq F(2\pi-0)$, then for sufficiently large values of $n$ all $a_n\neq 0$ and have the same sign and their absolute values decrease monotonously. If $F'(+0)\neq F'(2\pi-0)$ $F'(x)$ being the derivative of $F(x)$, a similar result holds for the coefficients $b_n$.

Here I will deal with such a problem more deeply.

Let $f(x)$ and all its derivatives be continuous and have limited total fluctuation in the interval $(0<x<2\pi)$ and have limits on the right and on the left at $x=0$ and at $x=2\pi$ respectively. Then the Fourier constants corresponding to the function $f(x)$ are

$$a_n=\frac{1}{\pi}\int_0^{2\pi}f(t)\sin nt\,dt,\quad b_n=\frac{1}{\pi}\int_0^{2\pi}f(t)\cos nt\,dt.$$

Hence

$$\frac{\pi}{2}\Delta_n=\int_0^{2\pi}\varphi(t)\cos\left(n+\frac{1}{2}\right)t\,dt,$$

where we have put

$$\Delta_n=a_{n+1}-a_n,\quad \varphi(t)=f(t)\sin\frac{t}{2}.$$

Integrating by parts we get

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(1) Fejér, Über die Fouriersche Reihe, Math. Ann., 64 (1907), p. 285

$$\frac{\pi}{2}\Delta_n = \frac{1}{2}\cdot\frac{1}{\left(n+\frac{1}{2}\right)^2}\left[f(2\pi-0)-f(+0)\right]$$

$$-\frac{1}{\left(n+\frac{1}{2}\right)^2}\int_0^{2\pi}\varphi''(t)\cos\left(n+\frac{1}{2}\right)t\,dt.$$

But by the fundamental lemma of Riemann-Lebesgue

$$\lim_{n\to\infty}\int_0^{2\pi}\varphi''(t)\cos\left(n+\frac{1}{2}\right)t\,dt=0;$$

so that for sufficiently large values of $n$

$$\Delta_n \gtrless 0 \quad \text{according as} \quad f(2\pi-0)\gtrless f(+0).$$

Since $\lim a_n=0$, it follows from the last inequalities that for sufficiently large values of $n$

$$a_n \gtrless 0 \quad \text{according as} \quad f(2\pi-0)\gtrless f(+0).$$

Also after a short calculation we can prove the inequalities (1):

$$\Delta_n \gtrless \Delta_{n+1} \quad \text{according as} \quad f(2\pi-0)\gtrless f(+0)\ (^2).$$

If $f(2\pi-0)=f(+0)$, then integrating by parts we have

$$\frac{\pi}{2}\Delta_n = -\frac{3}{2}\,\frac{1}{\left(n+\frac{1}{2}\right)^4}\left[f''(2\pi-0)-f''(+0)\right]$$

$$+\frac{1}{\left(n+\frac{1}{2}\right)^4}\int_0^{2\pi}\varphi^{(4)}(t)\cos\left(n+\frac{1}{2}\right)t\,dt;$$

so that

$$\Delta_n \gtrless 0,\ a_n \gtrless 0 \quad \text{according as} \quad f''(2\pi-0)\gtrless f''(+0).$$

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(1) In what follows the phrase "*for sufficiently large values of n*" will be omitted.

(2) Our inequalities for the Fourier constants have certain similarities with those for the Lebesgue constants

$$\rho_n = \frac{2}{\pi}\int_0^{\frac{\pi}{2}}\left|\frac{\sin(2n+1)t}{\sin t}\right|dt,$$

in which

$$\bar{\Delta}_n = \rho_{n+1}-\rho_n > 0, \qquad \bar{\Delta}_n > \bar{\Delta}_{n+1}.$$

See Gronwall, Über die Lebesgueschen Konstanten bei den Fourierschen Reihen Math .Ann., 72 (19.2), p. 244.

**2.** In general, when

$$f^{(2m)}(2\pi-0)=f^{(2m)}(+0) \quad (m=0, 1, 2, \cdots, p),$$

we have

$$\varphi^{(2p+3)}(2\pi-0)=-\frac{2p+3}{2}f^{(2p+2)}(2\pi-0),$$

$$\varphi^{(2p+3)}(+0)=\frac{2p+3}{2}f^{(2p+2)}(+0),$$

and

$$\frac{\pi}{2}\Delta_n=(-1)^{p+1}\frac{2p+3}{2}\frac{1}{\left(n+\frac{1}{2}\right)^{2p+4}}\left[f^{(2p+2)}(2\pi-0)-f^{(2p+2)}(+0)\right]$$

$$+(-1)^p\frac{1}{\left(n+\frac{1}{2}\right)^{2p+4}}\int_0^{2\pi}\varphi^{(2p+4)}(t)\cos\left(n+\frac{1}{2}\right)t\,dt.$$

Therefore we arrive at the theorem:

Theorem I.

(i) *If* $f(2\pi-0)\gtreqless f(+0)$, *then* $a_n\gtreqless 0$, $\Delta_n\gtreqless 0$, $\Delta_n\gtreqless\Delta_{n+1}$.

(ii) *If* $f^{(2m)}(2\pi-0)=f^{(2m)}(+0)$, $(m=0, 1, 2, \cdots, p)$

and

$$\begin{cases} f^{(2p+2)}(2\pi-0)>f^{(2p+2)}(+0), \\ \qquad\qquad \textit{then} \left\{a_n\gtreqless 0,\ \Delta_n\lesseqgtr 0,\ \Delta_n\lesseqgtr\Delta_{n+1}, \begin{matrix}\textit{for}\ p=0, 2, \cdots, \\ \textit{for}\ p=1, 3, \cdots,\end{matrix}\right. \\ f^{(2p+2)}(2\pi-0)<f^{(2p+2)}(+0), \\ \qquad\qquad \textit{then} \left\{a_n\lesseqgtr 0,\ \Delta_n\gtreqless 0,\ \Delta_n\gtreqless\Delta_{n+1}, \begin{matrix}\textit{for}\ p=0, 2, \cdots, \\ \textit{for}\ p=1, 3, \cdots,\end{matrix}\right. \end{cases}$$

From this theorem we can infer at once:

Theorem II. *For sufficiently large values of $n$, all the constants $a_n$ have a constant sign and their absolute values decrease monotonously, unless*

$$f^{(2p)}(2\pi-0)=f^{(2p)}(+0) \quad (p=0, 1, 2, \cdots, \infty).$$

In the particular case where $f(x)$ is analytic in the interval $(0\leqq x\leqq 2\pi)$, we can choose a fixed positive number $\rho$ such that

$$f(x)=f(0)+f'(0)\frac{x}{1!}+f''(0)\frac{x^2}{2!}+f'''(0)\frac{x^3}{3!}+\cdots\cdots, \quad |x|\leqq\rho,$$

$$f(x+2\pi)=f(2\pi)+f'(2\pi)\frac{x}{1!}+f''(2\pi)\frac{x^2}{2!}+f'''(2\pi)\frac{x^3}{3!}+\cdots\cdots,$$
$$|x|\leqq\rho.$$

Whence

$$f(x+2\pi)-f(x)=\sum_{p=0}^{\infty}\left[f^{(p)}(2\pi)-f^{(p)}(0)\right]\frac{x^p}{p!},\quad |x|\leqq\rho.$$

Therefore, *in order that however great $n$ may be, the constants $a_n$, corresponding to an analytic function $f(x)$ in the interval $(0\leqq x\leqq 2\pi)$, have no constant sign (or, become zero), $f(x+2\pi)-f(x)$ should be necessarily an odd function of $x$ (or identically zero)* (1).

**3.** Now we come to consider the coefficients $b_n$. If we put

$$\delta_n=b_{n+1}-b_n,$$

we have

$$\frac{\pi}{2}\delta_n=-\int_0^{2\pi}\varphi(t)\sin\left(n+\frac{1}{2}\right)t\,dt.$$

Integrating by parts we get

$$\frac{\pi}{2}\delta_n=-\frac{1}{\left(n+\frac{1}{2}\right)^3}\left[f'(2\pi-0)-f'(+0)\right]$$
$$+\frac{1}{\left(n+\frac{1}{2}\right)^3}\int_0^{2\pi}\varphi^{(3)}(t)\cos\left(n+\frac{1}{2}\right)t\,dt;$$

so that

$$\delta_n\lesseqgtr 0,\ b_n\gtreqless 0,\ \delta_n\lesseqgtr\delta_{n+1}\quad\text{according as}\quad f'(2\pi-0)\gtreqless f'(+0).$$

If $f'(2\pi-0)=f'(+0)$, then integrating by parts

$$\frac{\pi}{2}\delta_n=2.\frac{1}{\left(n+\frac{1}{2}\right)^5}\left[f'''(2\pi-0)-f'''(+0)\right]$$
$$-\frac{1}{\left(n+\frac{1}{2}\right)^5}\int_0^{2\pi}\varphi^{(5)}(t)\cos\left(n+\frac{1}{2}\right)t\,dt.$$

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(1) The converse of this theorem (or Theorem II) is not necessarily true. See Example IV in §4.

In general, when

$$f^{(2m-1)}(2\pi-0)=f^{(2m-1)}(+0), \quad (m=1, 2, \cdots, p),$$

we have

$$\varphi^{(2p+2)}(2\pi-0)=-(p+1)f^{(2p+1)}(2\pi-0),$$
$$\varphi^{(2p+2)}(+0)=(p+1)f^{(2p+1)}(+0),$$

and then

$$\frac{\pi}{2}\delta_n=(-1)^{p+1}(p+1)\cdot\frac{1}{\left(n+\frac{1}{2}\right)^{2p+3}}\left[f^{(2p+1)}(2\pi-0)-f^{(2p+1)}(+0)\right]$$
$$+(-1)^p\frac{1}{\left(n+\frac{1}{2}\right)^{2p+3}}\int_0^{2\pi}\varphi^{(2p+3)}(t)\cos\left(n+\frac{1}{2}\right)t\,dt.$$

Therefore we arrive at the theorems:

Theorem I′.

(i) *If* $f'(2\pi-0)\gtreqless f'(+0)$, *then* $b_n\gtreqless 0$, $\delta_n\lesseqgtr 0$, $\delta_n\lesseqgtr\delta_{n+1}$.

(ii) *If* $f^{(2m+1)}(2\pi-0)=f^{(2m+1)}(+0)$, $(m=0, 1, 2, \cdots, p)$

and

$$\begin{cases} f^{(2p+3)}(2\pi-0)>f^{(2p+3)}(+0), \\ \qquad\qquad \textit{then} \quad \{b_n\lesseqgtr 0,\ \delta_n\gtreqless 0,\ \delta_n\gtreqless\delta_{n+1} \ \begin{matrix}\textit{for} & p=0, 2, \cdots, \\ \textit{for} & p=1, 3, \cdots,\end{matrix} \\ f^{(2p+3)}(2\pi-0)<f^{(2p+3)}(+0), \\ \qquad\qquad \textit{then} \quad \{b_n\gtreqless 0,\ \delta_n\lesseqgtr 0,\ \delta_n\lesseqgtr\delta_{n+1} \ \begin{matrix}\textit{for} & p=0, 2, \cdots, \\ \textit{for} & p=1, 3, \cdots.\end{matrix} \end{cases}$$

Theorem II′. *For sufficiently large values of* $n$, *all the constants* $b_n$ *have a constant sign and their absolute values decrease monotonously, unless*

$$f^{(2p+1)}(2\pi+0)=f^{(2p+1)}(+0), \quad (p=0, 1, 2, \cdots, \infty).$$

Particularly, *in order that however great* $n$ *may be, the constants* $b_n$, *corresponding to an analytic function* $f(x)$ *in the interval* $(0\leqq x\leqq 2\pi)$, *have no constant sign (or, become zero),* $f(x+2\pi)-f(x)$ *should be necessarily an even function (or a constant)* (1).

Consequently we can infer the theorem immediately: *In order that however great* $n$ *may be, both* $a_n$ *and* $b_n$, *corresponding to an analytic function* $f(x)$ *in the interval* $(0\leqq x\leqq 2\pi)$, *have no constant signs (or, become zero), the function* $f(x)$ *should have the period* $2\pi$ (2).

(1), (2) The converse of these (and Theorem II′) is not necessarily true. See Example IV in §4.

**4.** Lastly we add some simple examples:

I. $$f(x)=e^x.$$

In this case

$$f(2\pi)>f(0),\quad f'(2\pi)>f'(0);$$

so that

$$a_n<0,\quad b_n>0.$$

In fact

$$e^x\sim\frac{e^{2\pi}-1}{\pi}\left[\frac{1}{2}-\frac{1}{1+1^2}\sin x+\frac{1}{1+1^2}\cos x-\frac{2}{1+2^2}\sin 2x\right.$$
$$\left.+\frac{1}{1+2^2}\cos 2x-\cdots\cdots-\frac{n}{1+n^2}\sin nx+\frac{1}{1+n^2}\cos nx-\cdots\cdots\right].$$

II. $$f(x)=e^x\sin x.$$

In this case

$$f(2\pi)=f(0),\quad f'(2\pi)>f'(0),\quad f''(2\pi)>f''(0);$$

so that

$$a_n>0,\quad b_n>0.$$

In fact

$$a_n=\frac{e^{2\pi}-1}{2\pi}\left[\frac{1}{1+(n-1)^2}-\frac{1}{1+(n+1)^2}\right]>0,\quad (n\geqq 1),$$

$$b_n=\frac{e^{2\pi}-1}{2\pi}\left[\frac{n+1}{1+(n-1)^2}-\frac{n+1}{1+(n+1)^2}\right]>0,\quad (n>1).$$

III. $$f(x)=-\frac{x}{2}.$$

In this case

$$f(2\pi)<f(0),\quad f^{(2p+1)}(2\pi)=f^{(2p+1)}(0),\quad (p=0,1,2,\cdots,\infty);$$

so that $a_n>0$ and $b_n$ have no constant sign (or, become zero). In fact

$$-\frac{x}{2}\sim\sin x+\frac{1}{2}\sin 2x+\frac{1}{3}\sin 3x+\cdots\cdots.$$

IV. Each of the following four functions has the period $2\pi$.

$$e^{\cos x}\cos(\sin x)\sim 1+\frac{1}{1!}\cos x+\frac{1}{2!}\cos 2x+\frac{1}{3!}\cos 3x+\cdots\cdots,$$

$$e^{-\cos x}\cos(\sin x)\sim 1-\frac{1}{1!}\cos x+\frac{1}{2!}\cos 2x-\frac{1}{3!}\cos 3x+\cdots\cdots,$$

$$e^{\cos x}\sin(\sin x)\sim \frac{1}{1!}\sin x+\frac{1}{2!}\sin 2x+\frac{1}{3!}\sin 3x+\cdots\cdots,$$

$$e^{-\cos x}\sin(\sin x)\sim \frac{1}{1!}\sin x+\frac{1}{2!}\sin 2x+\frac{1}{3!}\sin 3x-\cdots\cdots.$$

Also we have

$$\cosh(\cos x)\cos(\sin x)\sim 1+\frac{1}{2!}\cos 2x+\frac{1}{4!}\cos 4x+\cdots\cdots,$$

$$\begin{aligned} e^{\cos x}\cos(\sin x)+e^{-\cos x}\sin(\sin x)\sim 1&+\frac{1}{1!}\sin x-\frac{1}{2!}\sin 2x \\ &+\frac{1}{3!}\sin 3x-\cdots\cdots \\ &+\frac{1}{1!}\cos x+\frac{1}{2!}\sin 2x \\ &+\frac{1}{3!}\cos 3x+\cdots\cdots, \end{aligned}$$

etc.

## On the sine series.

**5.** Let $f(x)$ and all its derivatives be continuous and have limited total fluctuation in the interval $(0<x<\pi)$ and have limits on the right and on the left at $x=0$ and at $x=\pi$ respectively. If

$$\sum_{n=1}^{\infty} a_n \sin nx$$

be the sine series corresponding to the function $f(x)$, then

$$a_n=\frac{2}{\pi}\int_0^{\pi} f(t)\sin nt\,dt\,;$$

so that

$$\frac{\pi}{4}\Delta_n=\int_0^{\pi}\varphi(t)\cos\left(n+\frac{1}{2}\right)t\,dt,$$

where

$$\Delta_n = a_{n+1} - a_n, \quad \varphi(t) = f(t) \sin \frac{t}{2}.$$

Integrating by parts we get

$$\frac{\pi}{4} \Delta_n = (-1)^n \frac{1}{n+\frac{1}{2}} f(\pi - 0) - \frac{1}{n+\frac{1}{2}} \int_0^\pi \varphi'(t) \sin\left(n+\frac{1}{2}\right) t\, dt;$$

whence

$$\Delta_{2m} \gtrless 0, \ \Delta_{2m+1} \lessgtr 0 \quad \text{according as} \quad f(\pi - 0) \gtrless 0.$$

If $f(\pi - 0) = 0$, then

$$\frac{\pi}{4} \Delta_n = -\frac{1}{2} \frac{1}{\left(n+\frac{1}{2}\right)^2} f(+0) - \frac{1}{\left(n+\frac{1}{2}\right)^2} \int_0^\pi \varphi''(t) \cos\left(n+\frac{1}{2}\right) t\, dt;$$

so that

$$\Delta_n \lessgtr 0 \text{ (and consequently } a_n \gtrless 0) \text{ according as } f(+0) \gtrless 0.$$

Again if $f(\pi - 0) = 0$, $f(+0) = 0$, then

$$\frac{\pi}{4} \Delta_n = (-1)^{n+1} \frac{1}{\left(n+\frac{1}{2}\right)^3} f''(\pi - 0)$$

$$+ \frac{1}{\left(n+\frac{1}{2}\right)^3} \int_0^\pi \varphi^{(3)}(t) \sin\left(n+\frac{1}{2}\right) t\, dt;$$

whence

$$\Delta_{2m} \lessgtr 0, \ \Delta_{2m+1} \gtrless 0 \quad \text{according as} \quad f''(\pi - 0) \gtrless 0.$$

Further if $f(\pi - 0) = f(+0) = 0$, $f''(\pi - 0) = 0$, then

$$\frac{\pi}{4} \Delta_n = \frac{3}{2} \frac{1}{\left(n+\frac{1}{2}\right)^4} f''(+0) + \frac{1}{\left(n+\frac{1}{2}\right)^4} \int_0^\pi \varphi^{(4)}(t) \cos\left(n+\frac{1}{2}\right) t\, dt;$$

so that

$$\Delta_n \gtrless 0, \ a_n \lessgtr 0 \quad \text{according as} \quad f''(+0) \gtrless 0,$$

and so on. Thus we arrive at the theorem:

Theorem III.

(i) *If* $f(\pi-0)\gtreqless 0$, *then* $\Delta_{2m}\gtreqless 0$, $\Delta_{2m+1}\lesseqgtr 0$.

*If* $f(\pi-0)=0$ *and* $f(+0)\gtreqless 0$, *then* $\Delta_n\lesseqgtr 0$, $a_n\gtreqless 0$.

(ii) *If* $f^{(2\nu)}(\pi-0)=f^{(2\nu)}(+0)=0$, $(\nu=0, 1, 2, \cdots, p)$;

$$\text{and}\begin{cases} f^{(2p+2)}(\pi-0)>0, \text{ then } \{\Delta_{2m}\lesseqgtr 0,\ \Delta_{2m+1}\gtreqless 0, & \begin{matrix}\text{for } p=0, 2, \cdots, \\ \text{for } p=1, 3, \cdots,\end{matrix} \\ f^{(2p+2)}(\pi-0)<0, \text{ then } \{\Delta_{2m}\gtreqless 0,\ \Delta_{2m+1}\lesseqgtr 0, & \begin{matrix}\text{for } p=0, 2, \cdots, \\ \text{for } p=1, 3, \cdots.\end{matrix}\end{cases}$$

*If* $f^{(2\nu)}(\pi-0)=f^{(2\nu)}(+0)=0$, $(\nu=0, 1, 2, \cdots, p)$;
$f^{(2p+2)}(\pi-0)=0$,

$$\text{and}\begin{cases} f^{(2p+2)}(+0)>0, \text{ then } \{\Delta_n\gtreqless 0,\ a_n\lesseqgtr 0, & \begin{matrix}\text{for } p=0, 2, \cdots, \\ \text{for } p=1, 3, \cdots,\end{matrix} \\ f^{(2p+2)}(+0)<0, \text{ then } \{\Delta_n\lesseqgtr 0,\ a_n\gtreqless 0, & \begin{matrix}\text{for } p=0, 2, \cdots, \\ \text{for } p=1, 3, \cdots.\end{matrix}\end{cases}$$

**6.** Next we proceed to discuss the expression $\Delta_n+\Delta_{n+1}$. Since

$$\begin{aligned}\Delta_n+\Delta_{n+1}&=a_{n+2}-a_n\\ &=\frac{4}{\pi}\int_0^\pi \psi(t)\cos(n+1)t\,dt,\end{aligned}$$

where

$$\psi(t)=f(t)\sin t,$$

integrating by parts we have

$$\begin{aligned}\frac{\pi}{4}(a_{n+2}-a_n)&=\frac{2}{(n+1)^2}\Big[(-1)^n f(\pi-0)-f(+0)\Big]\\ &\quad-\frac{1}{(n+1)^2}\int_0^\pi \psi''(t)\cos(n+1)t\,dt.\end{aligned}$$

Hence

$a_{2m}\lesseqgtr a_{2m+2}$, (and consequently $a_{2m}\lesseqgtr 0$), according as
$$f(\pi-0)-f(+0)\gtreqless 0;$$

$a_{2m+1}\gtreqless a_{2m+3}$, (and consequently $a_{2m+1}\gtreqless 0$), according as
$$f(\pi+0)+f(+0)\gtreqless 0$$

I. If

$$f(\pi-0)-f(+0)=0,$$

then

$$\frac{\pi}{4}(a_{2m+2}-a_{2m})=-\frac{3}{(2m+1)^4}\left[f''(\pi-0)-f''(+0)\right]$$
$$+\frac{1}{(2m+1)^4}\int_0^\pi \varphi^{(4)}(t)\cos(2m+1)t\,dt;$$

so that

$$a_{2m}\gtreqless a_{2m+2},\quad a_{2m}\gtreqless 0,\quad \text{according as } f''(\pi-0)-f''(+0)\gtreqless 0.$$

Again if

$$f(\pi-0)=f(+0),\quad f''(\pi-0)=f''(+0),$$

then

$$\frac{\pi}{4}(a_{2m+2}-a_{2m})=\frac{5}{(2m+1)^6}\left[f^{(4)}(\pi-0)-f^{(4)}(+0)\right]$$
$$-\frac{1}{(2m+1)^6}\int_0^\pi \varphi^{(6)}(t)\cos(2m+1)t\,dt;$$

so that

$$a_{2m}\gtreqless a_{2m+2},\quad a_{2m}\gtreqless 0\quad \text{according as } f^{(4)}(\pi-0)-f^{(4)}(+0)\gtreqless 0;$$

and so on.

II. Similarly if

$$f(\pi-0)+f(+0)=0,$$

then

$$\frac{\pi}{4}(a_{2m+3}-a_{2m+1})=\frac{3}{(2m+2)^4}\left[f''(\pi-0)+f''(+0)\right]$$
$$+\frac{1}{(2m+2)^4}\int_0^\pi \varphi^{(4)}(t)\cos(2m+2)t\,dt;$$

so that

$$a_{2m+1}\gtreqless a_{2m+3},\quad a_{2m+1}\gtreqless 0\quad \text{according as}\quad f''(\pi-0)+f''(+0)\gtreqless 0;$$

and so on.

Therefore we can state the theorem:

Theorem IV.

I.

(i) *If* $f(\pi-0)-f(+0)\gtreqless 0$, *then* $a_{2m}\gtreqless 0$, $a_{2m}\gtreqless a_{2m+2}$.

(ii) *If* $f^{(2\nu)}(\pi-0)-f^{(2\nu)}(+0)=0, \quad (\nu=0, 1, 2, \cdots, p)$;

$$\text{and} \begin{cases} f^{(2p+2)}(\pi-0)-f^{(2p+2)}(+0)>0, \\ \qquad \text{then} \quad \{a_{2m} \gtreqless 0, \quad a_{2m} \gtreqless a_{2m+2}, \quad \begin{matrix} \text{for } p=0, 2, \cdots, \\ \text{for } p=1, 3, \cdots; \end{matrix} \\ f^{(2p+2)}(\pi-0)-f^{(2p+2)}(+0)<0, \\ \qquad \text{then} \quad \{a_{2m} \lesseqgtr 0, \quad a_{2m} \lesseqgtr a_{2m+2}, \quad \begin{matrix} \text{for } p=0, 2, \cdots, \\ \text{for } p=1, 3, \cdots. \end{matrix} \end{cases}$$

II.

(i) *If* $f(\pi-0)+f(+0) \gtreqless 0$, *then* $a_{2m+1} \gtreqless 0, \quad a_{2m+1} \gtreqless a_{2m+3}$.

(ii) *If* $f^{(2\nu)}(\pi-0)+f^{(2\nu)}(+0)=0, \quad (\nu=0, 1, 2, \cdots, p)$;

$$\text{and} \begin{cases} f^{(2p+2)}(\pi-0)+f^{(2p+2)}(+0)>0, \\ \qquad \text{then} \quad \{a_{2m+1} \lesseqgtr 0, \quad a_{2m+1} \lesseqgtr a_{2m+3}, \quad \begin{matrix} \text{for } p=0, 2, \cdots, \\ \text{for } p=1, 3, \cdots; \end{matrix} \\ f^{(2p+2)}(\pi-0)+f^{(2p+2)}(+0)<0, \\ \qquad \text{then} \quad \{a_{2m+1} \gtreqless 0, \quad a_{2m+1} \gtreqless a_{2m+3}, \quad \begin{matrix} \text{for } p=0, 2, \cdots, \\ \text{for } p=1, 3, \cdots \end{matrix} \end{cases}$$

It is apparent that this theorem contains some parts of Theorem III. Also from Theorem IV we can infer the following theorem immediately:

Theorem V. *For sufficiently large values of $m$, all the coefficients $a_{2m}$ of even terms (the coefficients $a_{2m+1}$ of odd terms) have a constant sign and their absolute values decrease monotonously, unless*

$$f^{(2p)}(\pi-0)-f^{(2p)}(+0)=0, \quad \{f^{(2p)}(\pi-0)+f^{(2p)}(+0)=0\}, \ (p=0, 1, 2, \cdots, \infty).$$

Therefore, *in order that however great $m$ may be, the coefficients $a_{2m}$ (or $a_{2m+1}$), corresponding to an analytic function $f(x)$ in the interval $(0 \leqq x \leqq \pi)$, have no constant sign or become zero, $f(x+\pi)-f(x)$ $\{$or $f(x+\pi)+f(x)\}$ should be necessarily an odd function of $x$ or identically zero* ([1]).

Also, *in order that however great $m$ may be, both $a_{2m}$ and $a_{2m+1}$, corresponding to an analytic function $f(x)$ in the interval $(0 \leqq x \leqq \pi)$, have no constant signs or become zero, both $f(x)$ and $f(x+\pi)$ should be necessarily odd functions of $x$.*

Lastly suppose that

$$f(\pi-0)>0, \quad f(\pi-0)-f(+0)>0.$$

Then

---

([1]) The converse of this theorem (or Theorem V) is not necessarily true. See Example VII in §7.

$$\begin{aligned} |\Delta_{2m+1}| - |\Delta_{2m}| &= -\Delta_{2m+1} - \Delta_{2m} \\ &= a_{2m} - a_{2m+2} \\ &< 0. \end{aligned}$$

In similar ways we can prove the theorem:

Theorem VI. *For sufficiently large values of $n$, there exist the following inequalities between the absolute values of $\Delta_n$ and $\Delta_{n+1}$:*

| | $f(\pi-0)>0,$ | $f(\pi-0)<0,$ |
|---|---|---|
| $f(\pi-0)-f(+0)\gtreqless 0,$ | $\lvert\Delta_{2m}\rvert \gtreqless \lvert\Delta_{2m+1}\rvert,$ | $\lvert\Delta_{2m}\rvert \lesseqgtr \lvert\Delta_{2m+1}\rvert,$ |
| $f(\pi-0)+f(+0)\gtreqless 0,$ | $\lvert\Delta_{2m+1}\rvert \gtreqless \lvert\Delta_{2m+2}\rvert,$ | $\lvert\Delta_{2m+1}\rvert \gtreqless \lvert\Delta_{2m+2}\rvert;$ |

*and so on.*

**7.** Here we add some examples:

I. $$f(x)=\frac{\pi-x}{2}.$$

In this case

$$f(\pi)=0, \quad f(0)>0;$$

so that

$$a_n>0.$$

In fact

$$\frac{\pi-x}{2} \sim \sin x + \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x + \cdots\cdots.$$

II. $$f(x)=\frac{x}{2}.$$

In this case

$$f(\pi)=\frac{\pi}{2}, \quad f(0)=0, \quad f(\pi)-f(0)>0, \quad f(\pi)+f(0)>0;$$

so that

$$a_{2m}<0, \quad a_{2m+1}>0, \quad |\Delta_{2m}|>|\Delta_{2m+1}|>|\Delta_{2m+2}|.$$

In fact

$$\frac{x}{2} \sim \sin x - \frac{1}{2} \sin 2x + \frac{1}{3} \sin 3x - \frac{1}{4} \sin 4x + \cdots\cdots.$$

III. The following example belongs to the same case as II.

$$\frac{\pi}{4} - \frac{\pi}{2} \cos x \sim \sin x - \frac{4}{3} \sin 2x + \frac{1}{3} \sin 3x - \frac{8}{15} \sin 4x$$

$$+ \cdots + \frac{1}{2m-1} \sin (2m-1)x - \frac{4m}{4m^2-1} \sin 2mx + \cdots\cdots,$$

in which we see that

$$|a_{2m-1}| < |a_{2m}|, \quad |\Delta_{2m}| > |\Delta_{2m+1}| > |\Delta_{2m+2}|.$$

IV. $$f(x) = \frac{\pi}{4} + \frac{\pi}{2} \cos x.$$

In this case

$$f(\pi) = -\frac{\pi}{4}, \quad f(0) = \frac{3}{4}\pi, \quad f(\pi) - f(0) < 0, \quad f(\pi) + f(0) > 0;$$

so that

$$a_{2m} > 0, \quad a_{2m+1} > 0, \quad |\Delta_{2m}| > |\Delta_{2m+1}|, \quad |\Delta_{2m+1}| < |\Delta_{2m+2}|.$$

In fact

$$\frac{\pi}{4} + \frac{\pi}{2} \cos x \sim \sin x + \frac{4}{3} \sin 2x + \frac{1}{3} \sin 3x + \cdots\cdots$$

$$+ \frac{1}{2m-1} \sin (2m-1)x + \frac{4m}{4m^2-1} \sin 2mx + \cdots,$$

where

$$|a_{2m-1}| < |a_{2m}|.$$

V. $$f(x) = \frac{\pi}{4}.$$

In this case

$$f(\pi) + f(0) > 0, \quad f^{(2p)}(\pi) - f^{(2p)}(0) = 0, \quad (p = 0, 1, 2, \cdots, \infty);$$

so that $a_{2m+1} > 0$ and $a_{2m}$ have no constant sign. In fact

$$\frac{\pi}{4} \sim \sin x + \frac{1}{3} \sin 3x + \frac{1}{5} \sin 5x + \cdots\cdots.$$

VI. $$f(x)=\cos(\cos x)\ \sinh(\sin x).$$

In this case $f(x)$ and $f(x+\pi)$ are odd functions of $x$. And

$$\cos(\cos x)\sinh(\sin x)\sim\frac{1}{1\,!}\sin x-\frac{1}{3\,!}\sin 3\,x+\frac{1}{5\,!}\sin 5\,x-\cdots\cdots.$$

VII. $$f(x)=e^{\cos x}\sin(\sin x).$$

In this case $f(x+\pi)+f(x)$ is an odd function, but

$$e^{\cos x}\sin(\sin x)\sim\frac{1}{1\,!}\sin x+\frac{1}{2\,!}\sin 2\,x+\frac{1}{3\,!}\sin 3\,x+\cdots\cdots.$$

## On the cosine series.

**8.** Let $f(x)$ be any function which satisfies the conditions in §5, and let

$$\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\cos n\,x$$

be the cosine series corresponding to that function. Then we can treat this series in similar ways as in the sine series; and the following are some typical results:

I.

(i) *If* $f'(\pi-0)\gtreqless 0$, *then* $\Delta_{2m}\lesseqgtr 0$, $\Delta_{2m+1}\gtreqless 0$.

*If* $f'(\pi-0)=0$, $f'(+0)\gtreqless 0$, *then* $\Delta_n\gtreqless 0$, $a_n\lesseqgtr 0$.

(ii) *If* $f^{(2\nu+1)}(\pi-0)=f^{(2\nu+1)}(+0)=0$, $(\nu=0, 1, 2, \cdots, p)$;

$$\text{and}\begin{cases} f^{(2p+3)}(\pi-0)>0, \text{ then } \{\Delta_{2m}\gtreqless 0,\ \Delta_{2m+1}\lesseqgtr 0, & \begin{matrix}\text{for } p=0, 2, \cdots,\\ \text{for } p=1, 3, \cdots;\end{matrix}\\ f^{(2p+3)}(\pi-0)<0, \text{ then } \{\Delta_{2m}\lesseqgtr 0,\ \Delta_{2m+1}\gtreqless 0, & \begin{matrix}\text{for } p=0, 2, \cdots,\\ \text{for } p=1, 3, \cdots.\end{matrix}\end{cases}$$

*If* $f^{(2\nu+1)}(\pi-0)=f^{(2\nu+1)}(+0)=0$, $(\nu=0, 1, 2, \cdots, p)$;
$f^{(2p+3)}(\pi-0)=0$,

$$\text{and}\begin{cases} f^{(2p+3)}(+0)>0, \text{ then } \{\Delta_n\lesseqgtr 0,\ a_n\gtreqless 0, & \begin{matrix}\text{for } p=0, 2, \cdots,\\ \text{for } p=1, 3, \cdots;\end{matrix}\\ f^{(2p+3)}(+0)<0, \text{ then } \{\Delta_n\gtreqless 0,\ a_n\lesseqgtr 0, & \begin{matrix}\text{for } p=0, 2, \cdots,\\ \text{for } p=1, 3, \cdots.\end{matrix}\end{cases}$$

II.

(i) *If* $f'(\pi-0)-f'(+0)\gtreqless 0$, *then* $a_{2m}\gtreqless 0$, $a_{2m}\gtreqless a_{2m+2}$.

(ii) *If* $f^{(2\nu+1)}(\pi-0)-f^{(2\nu+1)}(+0)=0$, $(\nu=0, 1, 2, \cdots, p)$;

$$\text{and}\begin{cases} f^{(2p+3)}(\pi-0)-f^{(2p+3)}(+0)>0, \\ \qquad\qquad \textit{then} \quad \{a_{2m}\lesseqgtr 0, \quad a_{2m}\lesseqgtr a_{2m+2}, \quad \begin{matrix}\textit{for} & p=0,2,\cdots; \\ \textit{for} & p=1,3,\cdots\end{matrix} \\ f^{(2p+2)}(\pi-0)-f^{(2p+3)}(+0)<0, \\ \qquad\qquad \textit{then} \quad \{a_{2m}\gtreqless 0, \quad a_{2m}\gtreqless a_{2m+2}, \quad \begin{matrix}\textit{for} & p=0,2,\cdots, \\ \textit{for} & p=1,3,\cdots.\end{matrix} \end{cases}$$

III.

(i) *If* $f'(\pi-0)+f'(+0)\gtreqless 0$, *then* $a_{2m+1}\lesseqgtr 0$, $a_{2m+1}\lesseqgtr a_{2m+3}$.

(ii) *If* $f^{(2\nu+1)}(\pi-0)+f^{(2\nu+1)}(+0)=0$, $(\nu=0, 1, 2, \cdots, p)$;

$$\text{and}\begin{cases} f^{(2p+3)}(\pi-0)+f^{(2p+3)}(+0)>0, \\ \qquad\qquad \textit{then} \quad \{a_{2m+1}\gtreqless 0, \quad a_{2m+1}\gtreqless a_{2m+3}, \quad \begin{matrix}\textit{for} & p=0,2,\cdots; \\ \textit{for} & p=1,3,\cdots;\end{matrix} \\ f^{(2p+3)}(\pi-0)+f^{(2p+3)}(+0)<0, \\ \qquad\qquad \textit{then} \quad \{a_{2m+1}\lesseqgtr 0, \quad a_{2m+1}\lesseqgtr a_{2m+3}, \quad \begin{matrix}\textit{for} & p=0,2,\cdots, \\ \textit{for} & p=1,3,\cdots.\end{matrix} \end{cases}$$

Ikeda near Ôsaka, June 1918.

# Trajectories in the Irreversible Field of Force on a Surface,

by

KINNOSUKE OGURA, Ôsaka.

Consider a dynamical system with two degrees of freedom which is reduced to the normal form ([1]):

$$\ddot{p}+\lambda\dot{q}=\frac{\partial\gamma}{\partial p}, \tag{1}$$

$$\ddot{q}-\lambda\dot{p}=\frac{\partial\gamma}{\partial q},$$

where $p$ and $q$ denote the two coordinates of the dynamical system, $\dot{p}$ and $\dot{q}$ their time derivatives, and $\lambda$, $\gamma$ are functions of $p$ and $q$. Then equations (1) admit the first integral

$$\frac{1}{2}(\dot{p}^2+\dot{q}^2)=\gamma+h, \tag{2}$$

$h$ being an arbitrary constant.

Throughout this paper I will consider the motion of a particle on a surface and confine myself to the case in which *$h$ has a definite value.* A few properties of the trajectories have been obtained in my previous paper ([2]); but in the present I will deal with certain properties of a different nature, some of which may be considered as generalizations of those for the conservative field (that is, for the reversible field) ([3]).

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([1]) Birkhoff, "Dynamical systems with two degrees of freedom," Trans. Amer. Math. Soc, 18 (1917), p. 204.

([2]) Ogura, "On a certain system of doubly infinite curves on a surface," Tôhoku Math. Journal, 8 (1915), p. 213; "A remark on the dynamical system with two degrees of freedom," Tôhoku Math. Journal, 15 (1919), p. 181. Hereafter these papers will be referred to as $O_1$ and $O_2$ respectively.

([3]) Ogura, "Trajectories in the conservative field of force," Part I, Tôhoku Math. Journal, 7 (1915), p. 123; Part II, Tôhoku Math. Journal, 9 (1916), p. 134; "On the striped net of curves without ambages in dynamics," Proc. Tôkyô Math.-Phys. Soc, II 9 (1918), p. 284; "Note Supplementary to the paper 'on the striped net of curves without ambages in dynamics'," Proc. Tôkyô Math.-Phys. Soc., II 9 (1918), p. 409. Hereafter these papers will be referred to as $O_3$, $O_4$, $O_5$, $O_6$ respectively.

# PART I. General Theorems.

## The condition for $\infty^1$ given orbits.

**1.** Let the linear element of a surface $S$ be

$$ds^2 = \mu\,(dp^2 + dq^2), \tag{3}$$

where isothermic parameters $p$ and $q$ are taken for parametrics. Then the differential equation of the orbits is (1)

$$q'' = \left(\frac{\partial \log \sqrt{\gamma+h}}{\partial q} - \frac{\partial \log \sqrt{\gamma+h}}{\partial p} q'\right)(1+q'^2) + \frac{\lambda}{\sqrt{2\,(\gamma+h)}}(1+q'^2)^{\frac{3}{2}}, \tag{4}$$

where we have put

$$q' = \frac{dq}{dp}, \qquad q'' = \frac{d^2q}{dp^2}.$$

Since $\gamma+h$ and $\mu$ are positive, we may put

$$\gamma + h = \frac{1}{2}\mu\,\varphi^2, \qquad \frac{\lambda}{\sqrt{2(\gamma+h)}} = \sqrt{\mu}\,\psi\,;$$

so that (4) becomes

$$q'' = \left(\frac{\partial \log (\sqrt{\mu}\,\varphi)}{\partial q} - \frac{\partial \log (\sqrt{\mu}\,\varphi)}{\partial p} q'\right)(1+q'^2) + \sqrt{\mu}\,\psi.\ (1+q'^2)^{\frac{3}{2}}. \tag{5}$$

The integral curves of this differential equation (2) will be called *the orbits in the field* $(\varphi, \psi)$.

Now suppose that

$$f(p, q) = \text{const.}$$

form a family of $\infty^1$ orbits in the field $(\varphi, \psi)$. Since

$$q' = -\left(\frac{\partial f}{\partial p}\right)\cdot\left(\frac{\partial f}{\partial q}\right)^{-1},$$

$$q'' = -\left[\left(\frac{\partial f}{\partial q}\right)^2 \frac{\partial^2 f}{\partial p^2} - 2\frac{\partial f}{\partial p}\,\frac{\partial f}{\partial q}\,\frac{\partial^2 f}{\partial p\,\partial q} + \left(\frac{\partial f}{\partial p}\right)^2 \frac{\partial^2 f}{\partial q^2}\right]\cdot\left(\frac{\partial f}{\partial q}\right)^{-3},$$

---

(1) See $O_2$.

(2) For some properties of the integral curves, see $O_1$.

we must have

$$\left[\left(\frac{\partial f}{\partial q}\right)^2\frac{\partial^2 f}{\partial p^2}-2\frac{\partial f}{\partial p}\frac{\partial f}{\partial q}\frac{\partial^2 f}{\partial p\,\partial q}+\left(\frac{\partial f}{\partial p}\right)^2\frac{\partial^2 f}{\partial q^2}\right]$$
$$+\frac{1}{2}\left(\frac{\partial \log\mu}{\partial p}\frac{\partial f}{\partial p}+\frac{\partial \log\mu}{\partial q}\frac{\partial f}{\partial q}\right)\left[\left(\frac{\partial f}{\partial p}\right)^2+\left(\frac{\partial f}{\partial q}\right)^2\right]$$
$$+\left(\frac{\partial \log\varphi}{\partial p}\frac{\partial f}{\partial p}+\frac{\partial \log\varphi}{\partial q}\frac{\partial f}{\partial q}\right)\left[\left(\frac{\partial f}{\partial p}\right)^2+\left(\frac{\partial f}{\partial q}\right)^2\right]$$
$$+\sqrt{\mu}\,\psi.\left[\left(\frac{\partial f}{\partial p}\right)^2+\left(\frac{\partial f}{\partial q}\right)^2\right]^{\frac{3}{2}}=0.$$

But since the expression of the geodesic curvature $\frac{1}{\rho_f}$ of $f=\text{const.}$:

$$-\frac{1}{\rho_f}=\frac{1}{\mu}\left\{\frac{\partial}{\partial p}\left(\sqrt{\mu}\frac{\partial f}{\partial p}\left[\left(\frac{\partial f}{\partial p}\right)^2+\left(\frac{\partial f}{\partial q}\right)^2\right]^{-\frac{1}{2}}\right)\right.$$
$$\left.+\frac{\partial}{\partial q}\left(\sqrt{\mu}\frac{\partial f}{\partial q}\left[\left(\frac{\partial f}{\partial p}\right)^2+\left(\frac{\partial f}{\partial q}\right)^2\right]^{-\frac{1}{2}}\right)\right\}$$

gives

$$\left(\frac{\partial f}{\partial q}\right)^2\frac{\partial^2 f}{\partial p^2}-2\frac{\partial f}{\partial p}\frac{\partial f}{\partial q}\frac{\partial^2 f}{\partial p\partial q}+\left(\frac{\partial f}{\partial p}\right)^2\frac{\partial^2 f}{\partial q^2}$$
$$+\frac{1}{2}\left(\frac{\partial \log\mu}{\partial p}\frac{\partial f}{\partial p}+\frac{\partial \log\mu}{\partial q}\frac{\partial f}{\partial q}\right)$$
$$=-\frac{\sqrt{\mu}}{\rho_f}\left[\left(\frac{\partial f}{\partial p}\right)^2+\left(\frac{\partial f}{\partial q}\right)^2\right]^{\frac{3}{2}};$$

whence we have

$$(6)\qquad \frac{1}{\rho_f}-\frac{1}{\sqrt{\mu}}\left(\frac{\partial \log\varphi}{\partial p}\frac{\partial f}{\partial p}+\frac{\partial \log\varphi}{\partial q}\frac{\partial f}{\partial q}\right)$$
$$\times\left[\left(\frac{\partial f}{\partial p}\right)^2+\left(\frac{\partial f}{\partial q}\right)^2\right]^{-\frac{1}{2}}-\psi=0.$$

Now consider the differential parameters of the differential quadratic form

$$E\,du^2+2\,F\,du\,dv+G\,dv^2:$$

namely

$$\Delta_1 U=\frac{E\left(\frac{\partial U}{\partial v}\right)^2-2F\frac{\partial U}{\partial u}\frac{\partial U}{\partial v}+G\left(\frac{\partial U}{\partial u}\right)^2}{EG-F^2},$$

$$\Delta_2 U=\frac{1}{\sqrt{EG-F^2}}\left\{\frac{\partial}{\partial u}\left[\left(G\frac{\partial U}{\partial u}-F\frac{\partial U}{\partial v}\right)\Big/\sqrt{EG-F^2}\right]\right.$$

$$\left.+\frac{\partial}{\partial v}\left[\left(E\frac{\partial U}{\partial v}-F\frac{\partial U}{\partial u}\right)\Big/\sqrt{EG-F^2}\right]\right\},$$

$$\Gamma(U, V)=\frac{E\frac{\partial U}{\partial v}\frac{\partial V}{\partial v}-F\left(\frac{\partial U}{\partial u}\frac{\partial V}{\partial v}+\frac{\partial U}{\partial v}\frac{\partial V}{\partial u}\right)+G\frac{\partial U}{\partial u}\frac{\partial V}{\partial u}}{EG-F^2}.$$

If we put

$$u=p, \qquad v=q;$$

$$E=\mu, \qquad F=0, \qquad G=\mu,$$

equation (6) may be written

$$(7) \qquad -\frac{1}{\rho_f}+\frac{1}{\sqrt{\Delta_1 f}}\,\Gamma(f, \log\varphi)+\psi=0,$$

or, by Beltrami's formula,

$$(8) \qquad \frac{\Delta_2 f}{\sqrt{\Delta_1 f}}+\Gamma\left(f, \frac{1}{\sqrt{\Delta_1 f}}\right)+\frac{1}{\sqrt{\Delta_1 f}}\Gamma(f, \log\varphi)+\psi=0.$$

This is the necessary and sufficient condition that $f(p, q)=\text{const.}$ should be orbits in the field $(\varphi, \psi)$ on the surface having the linear element

$$ds^2=\mu\,(dp^2+dq^2).$$

Let us now apply the transformation

$$p=p\,(u, v), \qquad q=q\,(u, v)$$

and let

$$\mu\,(dp^2+dq^2)=E\,du^2+2\,F\,du\,dv+G\,dv^2.$$

Putting

$$f\,(p, q)=\bar{f}\,(u, v), \qquad \varphi\,(p, q)=\bar{\varphi}\,(u, v), \qquad \psi\,(p, q)=\bar{\psi}\,(u, v)$$

and remembering the invariantive property of the differential parameters, (8) becomes

$$\frac{\Delta_2 \bar{f}}{\sqrt{\Delta_1 \bar{f}}}+\Gamma\left(\bar{f}, \frac{1}{\sqrt{\Delta_1 \bar{f}}}\right)+\frac{1}{\sqrt{\Delta_1 \bar{f}}}\Gamma\,(\bar{f}, \log\bar{\varphi})+\bar{\psi}=0.$$

Therefore we have the theorem:

*In order that $\infty^1$ curves*

$$f(u, v) = \text{const.}$$

*may be orbits in the field* $(\varphi(u, v), \psi(u, v))$ *on the surface having the linear element*

$$ds^2 = E\,du^2 + 2\,F\,du\,dv + G\,dv^2,$$

*it is necessary and sufficient that* $f(u, v)$ *should satisfy*

$$(8) \qquad \frac{\Delta_2 f}{\sqrt{\Delta_1 f}} + \Gamma\left(f, \frac{1}{\sqrt{\Delta_1 f}}\right) + \frac{1}{\sqrt{\Delta_1 f}}\Gamma(f, \log\varphi) + \psi = 0.$$

**2.** Let the two surfaces $S$, $\bar{S}$ having the linear elements

$$ds^2 = E\,du^2 + 2\,F\,du\,dv + G\,dv^2,$$
$$ds^2 = \bar{E}\,du^2 + 2\,\bar{F}\,du\,dv + \bar{G}\,dv^2$$

be related by the conformal representation such that

$$\frac{\bar{E}}{E} = \frac{\bar{F}}{F} = \frac{\bar{G}}{G} = \varphi^2(u, v).$$

Then the differential parameters corresponding to the surface $\bar{S}$ have the expressions:

$$\bar{\Delta}_1 f = \frac{1}{\varphi^2}\Delta_1 f, \qquad \bar{\Delta}_2 f = \frac{1}{\varphi^2}\Delta_2 f,$$

$$\bar{\Gamma}\left(f, \frac{1}{\sqrt{\bar{\Delta}_1 f}}\right) = \bar{\Gamma}\left(f, \frac{\varphi}{\sqrt{\Delta_1 f}}\right)$$
$$= \varphi.\,\bar{\Gamma}\left(f, \frac{1}{\sqrt{\Delta_1 f}}\right) + \frac{1}{\sqrt{\Delta_1 f}}.\,\bar{\Gamma}(f, \varphi)$$
$$= \frac{1}{\varphi}.\,\Gamma\left(f, \frac{1}{\sqrt{\Delta_1 f}}\right) + \frac{1}{\varphi\sqrt{\Delta_1 f}}\Gamma(f, \log\varphi);$$

so that (8) may be written

$$(9) \qquad \frac{\bar{\Delta}_2 f}{\sqrt{\Delta_1 f}}\bar{\Gamma} + \left(f, \frac{1}{\sqrt{\bar{\Delta}_1 f}}\right) + \frac{\psi}{\varphi} = 0.$$

Consequently if $\frac{1}{\rho_f}$ be the geodesic curvature of $f = \text{const.}$ on the surface $\bar{S}$, we have

$$\frac{1}{\bar{\rho}_f}=\frac{\psi}{\varphi}. \tag{10}$$

Thus we arrive at the theorem:

*By the conformal representation*

$$d\bar{s}^2=\varphi^2\,ds^2,$$

*the orbits* $f=$const. *in the field* $(\varphi, \psi)$ *on the surface* $S$ *are transformed into the curves having the geodesic curvature*

$$\frac{1}{\bar{\rho}_f}=\frac{\psi}{\varphi}$$

*on the surface* $\bar{S}$; *and conversely.*

In the particular case where $\varphi=$const., we have from (7)

$$\frac{1}{\rho_f}=\psi.$$

On the other hand, in the reversible field (that is, the case where $\psi=0$ identically), we have from (10)

$$\frac{1}{\bar{\rho}_f}=0;$$

so that $f=$const. are geodesics on the surface $\bar{S}$, which is a well known result ([1]).

## The condition for $2\infty^1$ given orbits.

**3.** Now we can infer from (8) the theorem immediately:

*The necessary and sufficient condition that the* $2\infty^1$ *curves* $u=$const., $v=$const. *may be orbits in the field* $(\varphi, \psi)$ *on the surface* $S$ *is given by*

$$G\frac{\partial \log \varphi}{\partial u}-F\frac{\partial \log \varphi}{\partial v}=\frac{EG-F^2}{G}\left\{\begin{matrix}2\,2\\1\end{matrix}\right\}-\sqrt{G}\sqrt{EG-F^2}\,\psi, \tag{11}$$

$$-F\frac{\partial \log \varphi}{\partial u}+E\frac{\partial \log \varphi}{\partial v}=\frac{EG-F^2}{E}\left\{\begin{matrix}1\,1\\2\end{matrix}\right\}-\sqrt{E}\sqrt{EG-F^2}\,\psi\ ([2]), \tag{12}$$

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([1]) Darboux, Théorie des surfaces, 2 (1. éd., 1889), p. 453; See also $O_3$, p. 178.

([2]) In the reversible field $\psi$ vanishes identically; so (11) and (12) become

$$G\frac{\partial \log \varphi}{\partial u}-F\frac{\partial \log \varphi}{\partial v}=\frac{EG-F^2}{G}\left\{\begin{matrix}2\,2\\1\end{matrix}\right\},$$

$$-F\frac{\partial \log \varphi}{\partial u}-E\frac{\partial \log \varphi}{\partial v}=\frac{EG-F^2}{E}\left\{\begin{matrix}1\,1\\2\end{matrix}\right\}$$

respectively. Compare with $O_3$ p. 179.

where $\left\{ {11 \atop 2} \right\}$, $\left\{ {22 \atop 1} \right\}$ denote Christoffel's symbols, that is,

$$\left\{ {11 \atop 2} \right\} = \frac{-F\dfrac{\partial E}{\partial u} - E\dfrac{\partial E}{\partial v} + 2E\dfrac{\partial F}{\partial u}}{2(EG-F^2)},$$

$$\left\{ {22 \atop 1} \right\} = \frac{-F\dfrac{\partial G}{\partial v} - G\dfrac{\partial G}{\partial u} + 2G\dfrac{\partial F}{\partial v}}{2(EG-F^2)}.$$

In order to interpret this condition geometrically, let us apply the conformal representation such that

$$d\bar{s}^2 = \varphi^2\, ds^2.$$

Then (11) and (12) are equivalent to

$$\frac{1}{\bar{\rho}_u} = \frac{\psi}{\varphi}, \qquad \frac{1}{\bar{\rho}_v} = \frac{\psi}{\varphi}$$

respectively (§2); from which we find

$$\frac{1}{\bar{\rho}_u} = \frac{1}{\bar{\rho}_v}. \tag{13}$$

Conversely, consider the parametric curves $u$, $v$ on a surface $S$, such that

$$\frac{1}{\bar{\rho}_u} = \frac{1}{\bar{\rho}_v} \; (= \Phi(u, v), \text{ say}).$$

Take any function $\varphi(u, v)$ and put $\psi = \varphi\Phi$. Then $u$, $v$ will be orbits in the field $(\varphi, \psi)$ on the surface $S$, which is obtained from $\bar{S}$ by the conformal transformation

$$\frac{1}{\varphi^2} d\bar{s}^2 = ds^2.$$

Thus we have the theorem:

*In order that $2\infty^1$ curves $u$, $v$ may be orbits on a surface $S$ in an irreversible field of force, it is necessary and sufficient that the corresponding curves on the surface $S$, obtained from $S$ by a certain conformal transformation, should have the property*

$$\frac{1}{\bar{\rho}_u} = \frac{1}{\bar{\rho}_v}.$$

**4.** Since (11), (12) may be written

$$(14)\qquad \frac{1}{\sqrt{EG-F^2}}\left[\sqrt{G}\,\frac{\partial \log \varphi}{\partial u}-\frac{F}{\sqrt{G}}\,\frac{\partial \log \varphi}{\partial v}\right]=\frac{1}{\rho_u}-\psi,$$

$$(15)\qquad \frac{1}{\sqrt{EG-F^2}}\left[-\frac{F}{\sqrt{E}}\,\frac{\partial \log \varphi}{\partial u}+\sqrt{E}\,\frac{\partial \log \varphi}{\partial v}\right]=\frac{1}{\rho_v}-\psi$$

respectively, if we put

$$F_u=\frac{\varphi^2}{\sqrt{G}}\,\frac{\partial \log \varphi}{\partial v},\qquad F_v=\frac{\varphi^2}{\sqrt{E}}\,\frac{\partial \log \varphi}{\partial u},$$

$$F_{u'}=\frac{\varphi^2}{\rho_u}-\varphi^2\psi,\qquad F_{v'}=\frac{\varphi^2}{\rho_v}-\varphi^2\psi.$$

we can derive the two equations:

$$(16)\qquad F_u\cos\frac{\omega}{2}+F_{u'}\sin\frac{\omega}{2}=F_v\cos\frac{\omega}{2}+F_{v'}\sin\frac{\omega}{2},$$

$$(17)\qquad F_u\sin\frac{\omega}{2}-F_{u'}\cos\frac{\omega}{2}=-F_v\sin\frac{\omega}{2}+F_{v'}\cos\frac{\omega}{2},$$

$\omega$ being the angle between $u=$const. and $v=$const..

Now denote by $\mathfrak{F}_u$ the vector having $F_u$ as its magnitude and the (positive) direction of the curve $u=$const. as its direction, and by $\mathfrak{F}_{u'}$ the vector having $F_{u'}$ as its magnitude and the direction to the centre of geodesic curvature of $u=$const. as its direction; also define $\mathfrak{F}_v$ and $\mathfrak{F}_{v'}$ in similar ways with respect to $v=$const.. Then it follows from (16) and (17) that the projections of the resultant of $\mathfrak{F}_u$ and $\mathfrak{F}_{u'}$ upon the two bisectors of the angles (internal and external) between $u=$const., $v=$const. are equal to those for $\mathfrak{F}_v$ and $\mathfrak{F}_{v'}$ upon the same bisectors respectively.

Consequently we arrive at the *dynamical interpretation* of the conditions (11) and (12):

*A necessary and sufficient condition that* $u=$const. *and* $v=$const. *may be orbits in the field* $(\varphi, \psi)$ *is that the resultant of the two vectors* $\mathfrak{F}_u$ *and* $\mathfrak{F}_{u'}$ *is equal to the resultant of* $\mathfrak{F}_v$ *and* $\mathfrak{F}_{v'}$ (1).

In the reversible field ($\psi=0$), if $\mathfrak{F}_{uo}$ ($\mathfrak{F}_{vo}$) and $\mathfrak{F}_{u'o}$ ($\mathfrak{F}_{v'o}$) be the components of force tangential and normal to the orbit $u=$const. ($v=$const.) respectively, we have the well known relations:

$$F_{u0}=\frac{\varphi^2}{\sqrt{G}}\,\frac{\partial \log \varphi}{\partial v},\qquad F_{v0}=\frac{\varphi^2}{\sqrt{E}}\,\frac{\partial \log \varphi}{\partial u},$$

(1) Compare with $O_4$, p. 140.

$$F_{w'0}=\frac{\varphi^2}{\rho_u}, \qquad F_{v'0}=\frac{\varphi^2}{\rho_v}.$$

**5.** In this paragraph I will give three pairs of the formulae concerning the geodesic curvatures of 2 $\infty^1$ orbits $u=$const., $v=$const. and their related curves.

I. Let $ds(u_1)$ and $ds(v_1)$ be the arc elements of the curves $u_1=$const. and $v_1=$const. which are the bisectors of the angles, external, between $u=$const. and $v=$const. respectively. Then

$$(18)\begin{cases} 2\sin\dfrac{\omega}{2}\cdot\dfrac{\partial}{\partial s(u_1)}(\log\varphi)=\dfrac{1}{\sqrt{E}}\dfrac{\partial}{\partial u}(\log\varphi)-\dfrac{1}{\sqrt{G}}\dfrac{\partial}{\partial v}(\log\varphi), \\ 2\cos\dfrac{\omega}{2}\cdot\dfrac{\partial}{\partial s(v_1)}(\log\varphi)=\dfrac{1}{\sqrt{E}}\dfrac{\partial}{\partial u}(\log\varphi)+\dfrac{1}{\sqrt{G}}\dfrac{\partial}{\partial v}(\log\varphi). \end{cases}$$

Therefore by means of (18) we find from (14) and (15) the following *formulae* of importance:

$$(19)\begin{cases} \cos\dfrac{\omega}{2}\cdot\dfrac{\partial}{\partial s(u_1)}(\log\varphi)=\dfrac{1}{2}\left(\dfrac{1}{\rho_u}-\dfrac{1}{\rho_v}\right), \\ \sin\dfrac{\omega}{2}\cdot\dfrac{\partial}{\partial s(v_1)}(\log\varphi)=\dfrac{1}{2}\left(\dfrac{1}{\rho_u}+\dfrac{1}{\rho_v}\right)-\psi. \end{cases}$$

II. Next suppose that $u'=$const., $v'=$const. are the orthogonal trajectories of $u=$const., $v=$const. respectively. Then we have

$$(20)\begin{cases} \dfrac{G}{E}\left\{\begin{matrix}11\\2\end{matrix}\right\}+\dfrac{F}{G}\left\{\begin{matrix}22\\1\end{matrix}\right\}=\dfrac{\sqrt{G}}{\rho_{w'}}-\dfrac{G}{\sqrt{EG-F^2}}\dfrac{\partial\omega}{\partial u}, \\ \dfrac{F}{E}\left\{\begin{matrix}11\\2\end{matrix}\right\}+\dfrac{E}{G}\left\{\begin{matrix}22\\1\end{matrix}\right\}=\dfrac{\sqrt{E}}{\rho_{v'}}-\dfrac{E}{\sqrt{EG-F^2}}\dfrac{\partial\omega}{\partial v}\ (^1). \end{cases}$$

Now from (11), (12) and (20) we get

$$\frac{1}{\sqrt{G}}\frac{\partial\log\varphi}{\partial v}=\frac{1}{\rho_{w'}}-\frac{\sqrt{G}}{\sqrt{EG-F^2}}\frac{\partial\omega}{\partial u}-\frac{\sqrt{EG}+F}{\sqrt{EG-F^2}}\psi,$$

$$\frac{1}{\sqrt{E}}\frac{\partial\log\varphi}{\partial u}=\frac{1}{\rho_{v'}}-\frac{\sqrt{E}}{\sqrt{EG-F^2}}\frac{\partial\omega}{\partial v}-\frac{\sqrt{EG}+F}{\sqrt{EG-F^2}}\psi.$$

Applying (18) to these two equations we find the *formulae:*

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(1) For an application of these formulae, see Ogura, "On a generalization of the Bonnet-Darboux theorem concerning the line of striction," Proc. Tôkyô Math.-Phys. Soc., II, 9 (1918), p. 304; where these formulae have been printed incorrectly.

$$(21)\begin{cases}\sin\dfrac{\omega}{2}\cdot\dfrac{\partial}{\partial s(u_1)}(\log\varphi)-\dfrac{1}{2}\sec\dfrac{\omega}{2}\cdot\dfrac{\partial\omega}{\partial s(u_1)}=-\dfrac{1}{2}\left(\dfrac{1}{\rho_{u'}}-\dfrac{1}{\rho_{v'}}\right),\\ \cos\dfrac{\omega}{2}\cdot\dfrac{\partial}{\partial s(v_1)}(\log\varphi)+\dfrac{1}{2}\operatorname{cosec}\dfrac{\omega}{2}\cdot\dfrac{\partial\omega}{\partial s(v_1)}=\dfrac{1}{2}\left(\dfrac{1}{\rho_{u'}}+\dfrac{1}{\rho_{v'}}\right)\\ \qquad\qquad -\cot\dfrac{\omega}{2}\cdot\psi.\end{cases}$$

III. Lastly applying the following formulae of Prof. Lilienthal [1]

$$\frac{2}{\rho(u_1)}=\sin\frac{\omega}{2}\cdot\left(\frac{1}{\rho_u}+\frac{1}{\rho_v}\right)+\cos\frac{\omega}{2}\cdot\left(\frac{1}{\rho_{u'}}+\frac{1}{\rho_{v'}}\right),$$

$$\frac{2}{\rho(v_1)}=-\cos\frac{\omega}{2}\cdot\left(\frac{1}{\rho_u}-\frac{1}{\rho_v}\right)+\sin\frac{\omega}{2}\cdot\left(\frac{1}{\rho_{u'}}-\frac{1}{\rho_{v'}}\right)$$

to (19) and (21), we obtain the *formulae:*

$$(22)\begin{cases}\dfrac{\partial}{\partial s(u_1)}(\log\varphi)-\dfrac{1}{2}\cot\dfrac{\omega}{2}\cdot\dfrac{\partial\omega}{\partial s(u_1)}=-\dfrac{1}{\rho(v_1)},\\ \dfrac{\partial}{\partial s(v_1)}(\log\varphi)+\dfrac{1}{2}\cot\dfrac{\omega}{2}\cdot\dfrac{\partial\omega}{\partial s(v_1)}=\dfrac{1}{\rho(u_1)}-\operatorname{cosec}\dfrac{\omega}{2}\cdot\psi.\end{cases}$$

## PART II. Some Particular Cases.

### 2 $\infty^1$ orbits comprising a constant angle.

**6.** Hereafter we will consider some system of 2 $\infty^1$ orbits in the field $(\varphi, \psi)$, which are of geometrical interest.

We begin with the case where *the angle $\omega$ between $u$=const., $v$=const. is constant.*

In this case (19), (21) and (22) give

$$\cos\frac{\omega}{2}\cdot\frac{\partial}{\partial s(u_1)}(\log\varphi)=\frac{1}{2}\left(\frac{1}{\rho_u}-\frac{1}{\rho_v}\right),$$

$$\sin\frac{\omega}{2}\cdot\frac{\partial}{\partial s(u_1)}(\log\varphi)=-\frac{1}{2}\left(\frac{1}{\rho_{u'}}-\frac{1}{\rho_{v'}}\right),$$

$$\frac{\partial}{\partial s(u_1)}(\log\varphi)=-\frac{1}{\rho(v_1)};$$

so that we find

[1] Lilienthal, Vorlesungen über Differentialgeometrie, II $_1$ (1913), pp. 240–241.

$$(23)\qquad \frac{2\cos\frac{\omega}{2}}{\rho(v_1)} = -\left(\frac{1}{\rho_u} - \frac{1}{\rho_v}\right),$$

$$(24)\qquad \frac{2\sin\frac{\omega}{2}}{\rho(v_1)} = \frac{1}{\rho_{u'}} - \frac{1}{\rho_{v'}}.$$

Similarly

$$(25)\qquad \frac{2\sin\frac{\omega}{2}}{\rho(u_1)} = \frac{1}{\rho_u} + \frac{1}{\rho_v},$$

$$(26)\qquad \frac{2\cos\frac{\omega}{2}}{\rho(u_1)} = \frac{1}{\rho_{u'}} + \frac{1}{\rho_{v'}}\ (^1).$$

Since the product of the right hand sides of (23) and (25) is equal to that for (24) and (26), we have

$$\frac{1}{\rho_u^2} + \frac{1}{\rho_{u'}^2} = \frac{1}{\rho_v^2} + \frac{1}{\rho_{v'}^2}.$$

Adding the squares of (23), (24), (25) and (26),

$$2\left(\frac{1}{\rho^2(u_1)} + \frac{1}{\rho^2(v_1)}\right) = \frac{1}{\rho_u^2} + \frac{1}{\rho_v^2} + \frac{1}{\rho_{u'}^2} + \frac{1}{\rho_{v'}^2}.$$

From the last two equations we obtain

$$\frac{1}{\rho^2(u_1)} + \frac{1}{\rho^2(v_1)} = \frac{1}{\rho_u^2} + \frac{1}{\rho_{u'}^2} = \frac{1}{\rho_v^2} + \frac{1}{\rho_{v'}^2}.$$

**7.** As a simple application, let us consider the $2\,\infty^1$ curves $u$, $v$ which make a constant angle and may be orbits in the irreversible field determined by

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(1) These four formulae have certain similarities with the following four concerning *a net of curves without ambages* $\bar{u} = \text{const.}$, $\bar{v} = \text{const.}$ :

$$\frac{4\cos\frac{\varpi}{2}}{\rho(\bar{v}_1)} = -\left(\frac{1}{\rho\bar{u}} - \frac{1}{\rho\bar{v}}\right), \qquad \frac{2\cos\varpi}{\rho(\bar{v}_1)} = -\sin\frac{\varpi}{2}\cdot\left(\frac{1}{\rho\bar{u}'} - \frac{1}{\rho\bar{v}'}\right),$$

$$\frac{4\sin\frac{\varpi}{2}}{\rho(\bar{u}_1)} = \frac{1}{\rho\bar{u}} + \frac{1}{\rho\bar{v}}, \qquad \frac{2\cos\varpi}{\rho(\bar{u}_1)} = \cos\frac{\varpi}{2}\cdot\left(\frac{1}{\rho\bar{u}'} + \frac{1}{\rho\bar{v}'}\right),$$

$\varpi$ being the angle between $\bar{u}$ and $\bar{v}$. See Lilienthal, loc. cit., pp 244-245.

$$\varphi = \text{const.}, \quad \varphi = \text{const.}.$$

From (19) we have

$$\frac{1}{\rho_u} = \frac{1}{\rho_v} = \text{const.}\ (= k, \text{ say});$$

so that from (23) and (25)

$$\frac{1}{\rho(v_1)} = 0, \qquad \frac{1}{\rho(u_1)} = \frac{k}{\sin\frac{\omega}{2}} = \text{const.}.$$

Consequently by well known theorems the surface must be a pseudospherical surface of revolution of the parabolic type or a surface applicable to it, and $u = \text{const.}$, $v = \text{const.}$ make constant angles ($\frac{\omega}{2}$ and $-\frac{\omega}{2}$ respectively) with the geodesic parallels $u_1 = \text{const.}$. Conversely if a surface have the linear element

$$ds^2 = du_1^2 + c^2 e^{\frac{2u_1}{a}} dv_1^2,$$

$a$, $c$ being constants, we have

$$\frac{1}{\rho(v_1)} = 0, \qquad \frac{1}{\rho(u_1)} = \frac{1}{a}.$$

Hence equations (22) are satisfied by

$$\omega = \text{const.}, \quad \varphi = \text{const.}, \quad \psi = \text{const.}, \quad \text{where } \operatorname{cosec}\frac{\omega}{2} \cdot \psi = \frac{1}{a}.$$

Therefore *in order that the* $2\,\infty^1$ *curves* $u = \text{const.}$, $v = \text{const.}$ *which make a constant angle and may be orbits in an irreversible field of the form* ($\varphi = \text{const.}$, $\psi = \text{const.}$), *it is necessary and sufficient that the surface should be a pseudospherical surface of revolution of the parabolic type or a surface applicable to it, and moreover the bisectors of the external angles of* $u$, $v$ *should be geodesic parallels.*

## The isothermal system as orbits.

**8.** For the reversible field of force (i. e. $\psi = 0$ identically), I have proved the theorem ([1]): A necessary and sufficient condition that $2\,\infty^1$ curves belonging to an orthogonal system may be orbits in a reversible

([1]) $O_1$, p 168, p. 173, p. 180.

field of force is that these curves form an isothermal system. Now I proceed to establish the following theorem concerning the irreversible field of force $(\varphi, \psi)$:

*Consider an orthogonal system* $u=\text{const.}$, $v=\text{const.}$. *If any three of the four propositions be true:*

(i) *that* $u$, $v$ *form an isothermal system;*

(ii) *that* $u$, $v$ *form a net of curves without ambages;*

(iii) *that* $\psi$ *is constant along each bisector* $u_1=\text{const.}$ *of the external angles of* $u$, $v$;

(iv) *that* $u$, $v$ *may be orbits in the field of the form* $(\varphi, \psi)$; *then the fourth is true also.*

In virtue of $F=0$, the necessary and sufficient conditions for (i), (ii), (iii), (iv) are given by

$$\text{(i)} \qquad \frac{\partial^2 \log E}{\partial u \partial v}=\frac{\partial^2 \log G}{\partial u \partial v};$$

$$\text{(ii)} \qquad \frac{\partial \sqrt{E}}{\partial v}=\frac{\partial \sqrt{G}}{\partial u} \text{ (1)};$$

$$\text{(iii)} \qquad \sqrt{G}\frac{\partial \psi}{\partial u}=\sqrt{E}\frac{\partial \psi}{\partial v};$$

$$\text{(iv)} \qquad \begin{cases} \dfrac{\partial \log \varphi}{\partial u}=-\dfrac{1}{2}\dfrac{\partial \log G}{\partial u}-\sqrt{E}\,.\,\psi, \\ \dfrac{\partial \log \varphi}{\partial v}=-\dfrac{1}{2}\dfrac{\partial \log E}{\partial v}-\sqrt{G}\,.\,\psi \end{cases}$$

respectively.

Firstly suppose that (i), (ii), (iii) are true. Then we have

$$\frac{1}{2}\frac{\partial^2 \log G}{\partial u \partial v}+\psi\frac{\partial \sqrt{E}}{\partial v}+\sqrt{E}\frac{\partial \psi}{\partial v}$$
$$=\frac{1}{2}\frac{\partial^2 \log E}{\partial u \partial v}+\psi\frac{\partial \sqrt{G}}{\partial u}+\sqrt{G}\frac{\partial \psi}{\partial u},$$

which shows us that

$$\left(\frac{1}{2}\frac{\partial \log G}{\partial u}+\sqrt{E}\,.\,\psi\right)du+\left(\frac{1}{2}\frac{\partial \log E}{\partial v}+\sqrt{G}\,.\,\psi\right)dv$$

(1) See, for example, R. Rothe, "Bemerkungen über die Gewebe (Kurvennetze ohne Umwege) auf einer Fläche," Jahresb. Deuts. Math. Ver., 17 (1903), p. 325.

is an exact differential. Hence if we define the function $\varphi$ by

$$\log\varphi=-\int\left(\frac{1}{2}\frac{\partial\log G}{\partial u}+\sqrt{E}\,.\,\psi\right)du+\left(\frac{1}{2}\frac{\partial\log E}{\partial v}+\sqrt{G}\,.\,\psi\right)dv,$$

equations (iv) are satisfied; that is, $u$, $v$ may be orbits in the field $(\varphi, \psi)$.

Next suppose that (i), (iii), (iv) are true. Then (iv) gives the condition of integrability

$$\frac{1}{2}\frac{\partial^2\log G}{\partial u\,\partial v}+\psi\frac{\partial\sqrt{E}}{\partial v}+\sqrt{E}\frac{\partial\psi}{\partial v}$$
$$=\frac{1}{2}\frac{\partial^2\log E}{\partial u\,\partial v}+\psi\frac{\partial\sqrt{G}}{\partial u}+\sqrt{G}\frac{\partial\psi}{\partial u},$$

from which we find, by means of (i) and (iii)

$$\frac{\partial\sqrt{E}}{\partial v}=\frac{\partial\sqrt{G}}{\partial u}.$$

In similar ways we can prove other two cases.

## The striped net of curves without ambages as orbits.

**9.** It is well known that $2\,\infty^1$ geodesics $u$, $v$ may be orbits in a reversible field when and only when $\varphi$ is constant ([1]). A generalization of this can be obtained from (19) immediately:

*If* $2\,\infty^1$ *curves* $u=$const., $v=$const. *such that*

$$\frac{1}{\rho_u}=\frac{1}{\rho_v}$$

*may be orbits in an irreversible field, the function* $\varphi$ *must be constant along* $u_1=$const.. *When the condition is fulfilled,* $\psi$ *is determined by*

$$\psi=\frac{1}{\rho}-\sin\frac{\omega}{2}\cdot\frac{\partial}{\partial s(v_1)}(\log\varphi),$$

where we have put

$$\frac{1}{\rho}=\frac{1}{\rho_u}=\frac{1}{\rho_v}.$$

Already I have dealt with the condition that a striped net of curves

([1]) $O_3$, p. 133, p. 181.

without ambages should be orbits in a reversible field ([1]). Now we can prove the theorem:

*Let* $u=\text{const.}$, $v=\text{const.}$ *be a striped net of curves without ambages. If any two of the three propositions be true:*

(i) *that each of* $u_1=\text{const.}$ *has a constant geodesic curvature;*

(ii) *that* $\psi$ *is constant along* $u_1=\text{const.}$;

(iii) *that* $u$, $v$ *may be orbits in the field of the form* $(\varphi, \psi)$;

*then the third is true also; and* $\varphi$ *is constant along* $u_1=\text{const.}$

I. Firstly we will deduce (iii) from (i) and (ii). Since $u$, $v$ form a striped net of curves without ambages, we may take

$$(27) \qquad E=\left(\frac{\partial u_1}{\partial u}\right)^2, \qquad F=\frac{\partial u_1}{\partial u}\frac{\partial u_1}{\partial v}\cos\omega, \qquad G=\left(\frac{\partial u_1}{\partial v}\right),$$

where $\omega$ is a function of $u_1$ alone and $u_1=\text{const.}$ are the bisectors of the external angles between $u$ and $v$ ([2]). Moreover we have

$$(28) \qquad \rho_u=\rho_v\,(=\rho,\ \text{say})\ ([3]).$$

But I have proved the following *lemma* ([4]): When $u$, $v$ form a striped net of curves without ambages, each of $u_1=\text{const.}$ has a constant geodesic curvature $\dfrac{1}{\rho(u_1)}$ when and only when $\rho$ is constant along $u_1=\text{const.}$

Hence it follows from (i) that $\rho$ is a function of $u_1$ only. Moreover we see from (ii) that $\psi$ is also a function of $u_1$ only. If we define the function $\varphi$ by

$$\log\varphi=\int\cot\frac{\omega}{2}\cdot\left(\frac{1}{\rho}-\psi\right)du_1,$$

then

$$\frac{\partial\log\varphi}{\partial u}=\cot\frac{\omega}{2}\cdot\left(\frac{1}{\rho}-\psi\right)\frac{\partial u_1}{\partial u},$$

$$\frac{\partial\log\varphi}{\partial v}=\cot\frac{\omega}{2}\cdot\left(\frac{1}{\rho}-\psi\right)\frac{\partial u_1}{\partial v}.$$

Consequently we have from (27), (28) and the last two equations

$$(14) \qquad \frac{1}{\sqrt{EG-F^2}}\left[\sqrt{G}\,\frac{\partial\log\varphi}{\partial u}-\frac{F}{\sqrt{G}}\,\frac{\partial\log\varphi}{\partial v}\right]=\frac{1}{\rho_u}-\psi,$$

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([1]) $O_5$, pp. 284–286; consult with $O_6$ also.

([2]), ([3]) Rothe, loc. cit.

([4]) See $O_6$.

$$(15)\qquad \frac{1}{\sqrt{EG-F^2}}\left[-\frac{F}{\sqrt{E}}\frac{\partial \log \varphi}{\partial u}+\sqrt{E}\,\frac{\partial \log \varphi}{\partial v}\right]=\frac{1}{\rho_v}-\psi.$$

Thus the proposition (iii) has been obtained.

II. Next we will deduce (ii) from (i) and (iii). Since $u$, $v$ form a striped net of curves without ambages, (28) must hold. Hence in order that (iii) is true, by the first theorem in this paragraph, $\varphi$ must be a function of $u_1$ only. But from (27) and (iii) we have

$$\frac{1}{\sqrt{EG-F^2}}\left[\sqrt{G}\,\frac{\partial \log \varphi}{\partial u}-\frac{F}{\sqrt{G}}\,\frac{\partial \log \varphi}{\partial v}\right]=\operatorname{tg}\frac{\omega(u_1)}{2}\cdot\frac{d}{du_1}\log\varphi(u_1)$$
$$=\frac{1}{\rho}-\varphi.$$

Moreover we see from (i) and the lemma above mentioned that $\rho$ is a function of $u_1$ only; so that $\psi$ must be a function of $u_1$ only.

III. Lastly we will deduce (i) from (ii) and (iii). From (iii) we get

$$\operatorname{tg}\frac{\omega(u_1)}{2}\cdot\frac{1}{du_1}\log\varphi(u_1)=\frac{1}{\rho}-\psi.$$

But by (ii) $\psi$ is a function of $u_1$ only; so that $\rho$ must be a function of $u_1$ alone. Therefore by the above lemma, each of $u_1=$const. has a constant geodesic curvature.

**10.** Now we see, from the last theorem, that if a striped net of curves without ambages $u=$const., $v=$const. may be orbits in the field $(\varphi, \psi(u_1))$, then each of $u_1=$const. should have a constant geodesic curvature. On the other hand, since $u$, $v$ form a striped net of curves without ambages, $u_1=$const. are geodesic parallels ([1]). Hence it follows that the linear element must take the form

$$ds^2=du_1^2+U_1(u_1)\,dv_1^2,$$

$U_1(u_1)$ being a function of $u_1$ alone; and the surface should be applicable to a surface of revolution.

Conversely, if a surface be applicable to a surface of revolution and $u$, $v$ form a striped net of curves without ambages, then we must have

$$ds^2=\sec^2\frac{\omega}{2}du_1^2+\mu^2\sin^2\frac{\omega}{2}dv_1^2$$
$$=\sec^2\frac{\omega}{2}du_1^2+f_1(u_1)f_2(v_1)\,dv_1^2,$$

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([1]) Rothe, loc. cit.

where $\omega$ and $f_1(u_1)$ are functions of $u_1$ alone and $f_2(v_1)$ of $v_1$ alone; so that $\mu$ takes the form $F_1(u_1)\,F_2(v_1)$. But in virtue of the formula

$$-\frac{2}{\rho(u_1)}=\sin\omega\,.\,\frac{\partial}{\partial u_1}\log\left(\mu\sin^2\frac{\omega}{2}\right),^{(1)}$$

$\rho(u_1)$ should be a function of $u_1$ alone. Therefore, by the last theorem, $u$, $v$ may be orbits in the field of the form $(\varphi, \psi(u_1))$. Thus we have theorem:

*In order that a surface may have a striped net of curves without ambages $u$, $v$ which can be orbits in the field $(\varphi, \psi(u_1))$, it is necessary and sufficient that the surface should be applicable to a surface of revolution.*

Moreover we can state the following theorem [2]:

*In order that $u$, $v$ constitute a striped net of curves without ambages which can be orbits in the field of the form $(\varphi, \psi(u_1))$, it is necessary and sufficient that the linear element should take the form*

$$ds^2=\xi(u+v)\Big[du^2+2\,\eta(u+v)\,du\,dv+dv^2\Big],$$

*where $\xi$ and $\eta$ are functions of $u+v$ only.*

Lastly we add the theorem:

*If a striped net of curves without ambages $u$, $v$ be orbits in the reversible field $(\varphi, 0)$, then the net may be orbits in the irreversible field of the form $(\varphi, \psi(u_1))$ also.*

Ikeda near Ôsaka, June 1918.

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(1) Rothe, loc. cit.

(2) For the method of proof, see $O_5$, pp. 287–288.

# A Theorem on Power Series,

by

YOSHITOMO OKADA, Ôsaka.

Dr. G. Pólya (1) has proved the theorem: If all the segments of a power series of $x$ which is formally written have real roots only, then the power series converges for every $x$ and represents a transcendental integral function of $x$ whose class is at most unity.

Modifying this slightly, we will prove the following

*Theorem: If the roots of all the segments of a power series of $x$ which is formally written do not lie on the origin side of a straight line not passing through the origin, then the power series represents a transcendental integral function of $x$ whose class is at most unity.*

Proof. Let a power series which is formally written be

$$F(x) \equiv a_0 + a_1 x + a_2 x^2 + \cdots\cdots + a_{n-1} x^{n-1} + a_n x^n + \cdots\cdots,$$

then by the assumption, the roots of all the segments

$$S_n(x) \equiv a_0 + a_1 x + a_2 x^2 + \cdots\cdots + a_{n-1} x^{n-1} + a_n x^n$$

$$(n = 1, 2, 3, \cdots\cdots),$$

do not lie on the origin side of a straight line not passing through the origin. Therefore, by Gauss's theorem, the root of

$$\frac{d^{n-1} S_n(x)}{dx^{n-1}} = (n-1)!\,(a_{n-1} + n\, a_n\, x)$$

does not lie on the origin side of the straight line. Let the distance of the straight line from the origin be $p$ $(\neq 0)$.

Then it must be

$$|x| = \left| -\frac{a_{n-1}}{n\, a_n} \right| \geqq p,$$

i.e.

$$|a_{n-1}| \geqq n\, p\, |a_n|.$$

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(1) Rendiconti del Circolo Matematico di Palermo, t. 36, 1913, pp. 279-295. See also the theorem II in E. Lindwart and G. Pólya's paper in the same Journal t. 37, 1914, p. 299.

Putting $n-1$, $n-2$, $\cdots\cdots$, 2, 1 instead of $n$, and multiplying up the results

$$|a_0| \geqq n!\, p^n |a_n|,$$

i.e.

$$|a_n| \leqq \frac{(p^{-1})^n |a_0|}{n!}.$$

Therefore $$|F(x)| \leqq |a_0| \exp.\, p^{-1} |x|.$$

This inequality shows that $F(x)$ is a transcendental integral function of $x$ whose class is at most unity.

# A Theorem on Limits,

by

YOSHITOMO OKADA, Ôsaka.

The following theorem was proved by some mathematicians (1).

If the sequence $x_1, x_2, x_3, \cdots\cdots, x_n, \cdots\cdots$ be of such a nature that

$$\lim_{n\to\infty}\left(x_n+q\frac{x_1+x_2\cdots\cdots+x_n}{n}\right)=0,$$

then

$$\lim_{n\to\infty} x_n=0,$$

provided that $1+q$ is positive; but when $1+q$ is not positive, this theorem is, in general, untrue.

Recently Messrs. T. Kojima (2) and S. Narumi (3) obtained some extended theorems respectively.

In this paper, I will prove, after the method of Prof. W. Sierpiński, another extended theorem:

*Theorem. Let*

$$\sum_{n\to 2}^{\infty} k_n/(k_1+k_2+\cdots\cdots+k_{n-1}) \tag{1}$$

*be a divergent series of positive terms, and if the complex sequence* $x_1, x_2, x_3, \cdots\cdots, x_n, \cdots\cdots$ *be of such a nature that*

$$\lim_{n\to\infty}\left(x_n+q\frac{k_1x_1+k_2x_2+\cdots\cdots+k_nx_n}{k_1+k_2+\cdots\cdots+k_n}\right)=l, \tag{2}$$

*$l$ being finite, then*

$$\lim_{n\to\infty} x_n=l/(1+q)$$

*provided that $1+q$ is positive; but when*

$$\varliminf_{n\to\infty}(k_1+k_2+\cdots\cdots+k_{n-1})/k_n>-(1+q)\geqq 0, \tag{4}$$

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(1) I. Schur, Math. Ann., Bd. 74, pp. 447-458.
Knopp, Math. Ann., Bd. 74, pp. 459-461.
W. Sierpiński, Tôhoku Math. Jour., Vol. 11, pp. 1-4.

(2) Tôhoku Math. Jour., Vol. 12, pp. 177-180, and pp. 323-326.

(3) Tôhoku Math. Jour., Vol. 12, pp. 275-290.

*the result is, in general, untrue.*

Proof. Put

$$k_1x_1+k_2x_2+\cdots\cdots+k_nx_n=\xi_n \quad (n=1, 2, 3, \cdots\cdots),$$

and

$$k_1+k_2+\cdots\cdots+k_n=\eta_n \quad (n=1, 2, 3, \cdots\cdots);$$

then we can rewrite (2) in the form

$$\lim_{n\to\infty}\left(\frac{\xi_n-\xi_{n-1}}{\eta_n-\eta_{n-1}}+q\frac{\xi_n}{\eta_n}\right)=l, \tag{5}$$

or

$$\frac{\xi_n-\xi_{n-1}}{\eta_n-\eta_{n-1}}+q\frac{\xi_n}{\eta_n}=l+\varepsilon_n, \quad (n=2, 3, 4 \cdots\cdots) \tag{6}$$

where

$$\lim_{n\to\infty}\varepsilon_n=0.$$

Put

$$\xi_n/\eta_n=y_n \quad (n=1, 2, 3, \cdots\cdots); \tag{7}$$

then from (6) and (7) we have

$$\left(1+q+\frac{\eta_{n-1}}{\eta_n-\eta_{n-1}}\right)y_n=\frac{\eta_{n-1}}{\eta_n-\eta_{n-1}}y_{n-1}+l+\varepsilon_n \quad (n=2, 3, 4, \cdots\cdots). \tag{8}$$

If we put again

$$u_{n-1}=\frac{\eta_{n-1}}{\eta_n-\eta_{n-1}}=(k_1+k_2+\cdots\cdots+k_{n-1})/k_n \quad (n=2, 3, 4, \cdots\cdots),$$

then (8) becomes

$$(1+q+u_{n-1})\,y_n=u_{n-1}y_{n-1}+l+\varepsilon_n, \quad (n=2, 3, 4, \cdots\cdots). \tag{9}$$

Consider the sequence $\{t_n\}$ defined by

$$t_n=\left(1+\frac{1+q}{u_1}\right)\left(1+\frac{1+q}{u_2}\right)\cdots\cdots\left(1+\frac{1+q}{u_n}\right) \quad (n=1, 2, 3, \cdots\cdots);$$

then by the assumptions that $1+q>0$ and that (1) is a divergent series of positive terms, we see that the sequence increases without limit monotonously, and

$$\frac{t_{n-1}}{t_{n-2}}=\frac{1+q+u_{n-1}}{u_{n-1}} \text{ and } t_{n-1}-t_{n-2}=\frac{1+q}{u_{n-1}}t_{n-2}, \quad (n=3, 4, 5, \cdots\cdots).$$

By substituting these values in (9)

$$t_{n-1}\left(y_n-\frac{l}{1+q}\right)=t_{n-2}\left(y_{n-1}-\frac{l}{1+q}\right)+\frac{t_{n-1}-t_{n-2}}{1+q}\varepsilon_n \quad (n=3, 4, 5, \cdots\cdots).$$

From these relations, we can prove in a way analogous to Sierpiński's that

$$\lim_{n\to\infty}\left(y_n-\frac{l}{1+q}\right)=0,$$

that is,

$$\lim_{n\to\infty} y_n=\lim_{n\to\infty} \xi_n/\eta_n=l/(1+q).$$

Accordingly by (5)

$$\lim_{n\to\infty}\frac{\xi_n-\xi_{n-1}}{\eta_n-\eta_{n-1}}=\lim_{n\to\infty} x_n=\frac{l}{1+q}.$$

If (4) be assumed, however, we can find a sequence $\{x_n\}$ which satisfies (2), but not (3).

For, by (4) there exists a fixed integer $m$, such that

$$(k_1+k_2+\cdots+k_{n-1})/k_n>-(1+q)\geqq 0, \quad \text{for } n\geqq m,$$

or

$$\eta_n+qk_m>0, \quad \text{for} \quad n\geqq m, \tag{10}$$

and

$$q\leqq -1. \tag{11}$$

Take the sequence $\{x_n\}$, such that

$$x_1=x_2=\cdots\cdots=x_{m-1}=0, \quad x_m=\eta_m/(\eta_m+qk_m), \tag{12}$$

and

$$x_n=\frac{k_m\eta_m}{\eta_m+q\,k_m}\cdot\frac{\eta_{m+1}}{\eta_{m+1}+q\,k_{m+1}}\cdots\cdots\frac{\eta_{n-1}}{\eta_{n-1}+qk_{n-1}}\cdot\frac{-q}{\eta_n+qk_n}, \quad \text{for } n>m; \tag{13}$$

then we see that

$$x_n + q\frac{k_1 x_1 + k_2 x_2 + \cdots\cdots + k_n x_n}{k_1 + k_2 + \cdots\cdots + k_n} = 0, \quad \text{for} \quad n > m,$$

which shows that the sequence $\{x_n\}$ satisfies (2). From (10) and (11), however, we have the inequalities

$$0 < \eta_{n+1} + qk_{n+1} \leqq \eta_n, \quad \text{for} \quad n \geqq m.$$

Hence by (13) we have

$$x_n \geqq \frac{k_m \eta_m}{\eta_m + qk_m} \cdot \frac{\eta_{m+1}}{\eta_m} \cdot \cdots\cdots \cdot \frac{\eta_{n-1}}{\eta_{n-2}} \cdot \frac{-q}{\eta_{n-1}} = \frac{-qk_m}{\eta_m + qk_m},$$

which shows that the sequence $\{x_n\}$ does not satisfy (3).

Thus our theorem is completely proved.

If we put

$$k_1 = 1 \text{ and } k_n/(k_1 + k_2 + \cdots + k_{n-1}) = a_{n-1}, \quad (n = 2, 3, 4, \cdots\cdots)$$

the above theorem is transformed into the following

*Corollay. Let*

$$\sum_{n \to 1}^{\infty}{}' a_n$$

*be a divergent series of positive terms, and if the complex sequence* $x_1, x_2, \cdots\cdots, x_n, \cdots\cdots$ *be of such a nature that*

$$\lim_{n \to \infty}\left(x_n + q\,\frac{x_1 + a_1 x_2 + a_2(1 + a_1)x_3 + \cdots\cdots + a_{n-1}(1 + a_1)(1 + a_2) \cdots\cdots (1 + a_{n-2})x_n}{(1 + a_1)(1 + a_2) \cdots\cdots (1 + a_{n-1})}\right)$$
$$= l,$$

*$l$ being finite, then*

$$\lim_{n \to \infty} x_n = l/(1 + q),$$

*provided that $1 + q$ is possitive; but when*

$$\varliminf_{n \to \infty} 1/a_n > -(1 + q) \geqq 0,$$

*the result is, in general, untrue.*

# An Extension of a Theorem of Scheeffer's,

by

YOSHIMICHI UCHIDA, Sendai.

Scheeffer has proved ([1]) the following theorem: If $G$ be a perfect set, nowhere dense, and $A$ a countable set, both lying on the same straight line, then for any given two numbers $c'$ and $c''$, there is at least one number $c$, lying between $c'$ and $c''$, both inclusive, such that if the set $A$ be translated by the distance $c$, no point of $A$ will coincide with a point of $G$.

This theorem can, by slightly modifying Scheeffer's proof, be extended to that in any dimensional space. I will next treat only the case for a plane set of points.

Let $A_1, A_2, \cdots\cdots$ be the points of a given countable set and $G$ be a given perfect, nowhere dense set, both lying on the same plane.

Take any point $A$, and draw a small circle with centre $A$, within which we can determine a square $p_1$, so that in it no point of $G$ falls. Let the distances of the sides of this square from $A_1$ be $a_1'$, $a_1''$, $b_1'$, $b_1''$ $(a_1''>a_1',\ b_1''>b_1')$.

Next take the square $p_1'$, the distances of whose sides from $A_2$ are $a_1'$, $a_1''$, $b_1'$, $b_1''$, direction being taken into account. Then in this square $p_1'$ we can take the second square $p_2$ having the same properties as $p_1$.

Let the distances of the sides of this square from $A_2$ be $a_2'$, $a_2''$, $b_2'$, $b_2''$ $(a_2''>a_2',\ b_2''>b_2')$; and so on.

Since the points $A_1, A_2, \cdots\cdots$ are countable, we have thus two sequences

$$a_1'<a_2'<a_3'<\cdots\cdots(\alpha)\cdots\cdots<a_3''<a_2''<a_1''$$

and

$$b_1'<b_2'<b_3'<\cdots\cdots(\beta)\cdots\cdots<b_3''<b_2''<b_1'',$$

from which at least two numbers $\alpha$, $\beta$ can be determined so that

$$a_r'\leqq\alpha\leqq a_r'',\ b_r'\leqq\beta\leqq b_r''\ (r=1, 2, \cdots\cdots).$$

The two numbers $\alpha$ and $\beta$ may lie between any two assigned numbers respectively.

If we now transfer the set $A$ by $x=\alpha$, $y=\beta$, then the two sets $A$ and $G$ have no common point clearly.

In the same way we can prove Scheeffer's theorem for more general cases.

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([2]) W. H. Young and G. C. Young, Theory of sets of points, 1906, p. 52.

# Über eine Ungleichung für bestimmte Integrale,

von

MATSUSABURÔ FUJIWARA in Sendai.

In meiner Arbeit: Ein von Brunn vermuteter Satz über konvexe Flächen und eine Verallgemeinerung der Schwarzschen und der Tchebycheffschen Ungleichungen für bestimmte Integrale, in diesem Journal, 14, habe ich einige Ungleichungen von der Form

$$\int_a^b f_1 \varphi_1 \, dx \int_a^b f_2 \varphi_2 \, dx \geqq \int_a^b f_1 \varphi_2 \, dx \int_a^b f_2 \varphi_1 \, dx$$

behandelt. Neulich bin ich darauf aufmerksam geworden, dass Herr Brunn (1) selbst auch sich mit der Tchebycheffschen Ungleichung

$$(b-a)\int_a^b f \varphi \, dx \geqq \int_a^b f \, dx \int_a^b \varphi \, dx$$

(nicht aber mit meiner Ungleichung) beschäftigt hatte. Seine Bedingung unter welcher die Ungleichung gültig bleibt, ist viel allgemeiner als diejenige von Herrn Franklin (2); damit hat Herr Brunn den Vorteil seiner Methode, in Vergleich mit der Franklinsche, wenn auch etwas umständlicher ist, überzeugt. Ich möchte aber bemerken, dass der Franklinsche Kunstgriff, d. h. die Integraldarstellung

$$(b-a)\int_a^b f \varphi \, dx - \int_a^b f \, dx \int_a^b \varphi \, dx = \frac{1}{2}\int_a^b \int_a^b (f(x)-f(y))\,(\varphi(x)-\varphi(y))\, dx\, dy$$

auch zur Herleitung des Brunnschen Resultat hinreicht, ja sogar einen Schritt weiter gehen kann.

Ich gehe wie Franklin von der folgenden Relation aus:

$$\int_a^b f_1 \varphi_1 \, dx \int_a^b f_2 \varphi_2 \, dx - \int_a^b f_1 \varphi_2 \, dx \int_a^b f_2 \varphi_1 \, dx$$

$$= \frac{1}{2}\int_a^b \int_a^b \begin{vmatrix} f_1(x) & f_1(y) \\ f_2(x) & f_2(y) \end{vmatrix} \cdot \begin{vmatrix} \varphi_1(x) & \varphi_1(y) \\ \varphi_2(x) & \varphi_2(y) \end{vmatrix} dx\, dy$$

$$= \frac{1}{2}\int_a^b \int_a^b f_2(x) f_2(y)\, \varphi_2(x)\, \varphi_2(y) \left( \frac{f_1(x)}{f_2(x)} - \frac{f_1(y)}{f_2(y)} \right) \left( \frac{\varphi_1(x)}{\varphi_2(x)} - \frac{\varphi_1(y)}{\varphi_2(y)} \right) dx\, dy.$$

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(1) Brunn: Neue Mittelwertssätze über bestimmte Integrale, Münchener Ber. 1902; und Nachtrag dazu, Münchener Ber. 1903.

(2) Franklin: Proof of a theorem of Tchebycheff's on definite integrals, Amer. Math. Journ. 7, 1885.

Ich transformiere dieses Doppelintegral $J$ in die Form

$$J=\int_a^b\int_a^b F(x,y)\left\{\left(\frac{f_1(x)}{f_2(x)}-A\right)-\left(\frac{f_1(y)}{f_2(y)}-A\right)\right\}$$
$$\times\left\{\left(\frac{\varphi_1(x)}{\varphi_2(x)}-B\right)-\left(\frac{\varphi_1(y)}{\varphi_1(y)}-B\right)\right\}dx\,dy,$$

wo $F(x,y)=f_2(x)f_2(y)\varphi_2(x)\varphi_2(y)$ und $A$, $B$ zwei Konstanten bedeuten. Dann wird

$$J=\int_a^b\int_a^b F(x,y)\left(\frac{f_1(x)}{f_2(x)}-A\right)\left(\frac{\varphi_1(x)}{\varphi_2(x)}-B\right)dx\,dy$$
$$+\int_a^b\int_a^b F(x,y)\left(\frac{f_1(y)}{f_2(y)}-A\right)\left(\frac{\varphi_1(y)}{\varphi_2(y)}-B\right)dx\,dy$$
$$-\int_a^b\int_a^b F(x,y)\left(\frac{f_1(x)}{f_2(x)}-A\right)\left(\frac{\varphi_1(y)}{\varphi_2(y)}-B\right)dx\,dy$$
$$-\int_a^b\int_a^b F(x,y)\left(\frac{f_1(y)}{f_2(y)}-A\right)\left(\frac{\varphi_1(x)}{\varphi_2(x)}-B\right)dx\,dy$$
$$=2\int_a^b\int_a^b F(x,y)\left(\frac{f_1(x)}{f_2(x)}-A\right)\left(\frac{\varphi_1(x)}{\varphi_2(x)}-B\right)dx\,dy$$
$$-2\int_a^b f_2(x)\,\varphi_2(x)\left(\frac{f_1(x)}{f_2(x)}-A\right)dx.\int_a^b f_2(y)\,\varphi_2(y)\left(\frac{\varphi_1(y)}{\varphi_2(y)}-B\right)dy.$$

Wenn man daher ausser $f_2(x)\,\varphi_2(x)>0$ noch die Relation

$$\frac{\varphi_1(x)}{\varphi_2(x)}\gtreqless B,\text{ je nachdem }\frac{f_1(x)}{f_2(x)}\gtreqless A,$$

voraussetzt, und ferner den Wert von $A$ so bestimmt, dass

$$\int_a^b f_2(x)\,\varphi_2(x)\left(\frac{f_1(x)}{f_2(x)}-A\right)dx=0,$$

d. h.

$$A=\int_a^b f_1\,\varphi_2\,dx:\int_b^b f_2\,\varphi_2\,dx$$

wird, so ergibt sich

$$J=2\int_a^b\int_a^b F(x,y)\left(\frac{f_1(x)}{f_2(x)}-A\right)\left(\frac{\varphi_1(x)}{\varphi_2(x)}-B\right)dx\,dy$$
$$\geqq 0,$$

worin das Gleichheitszeichen gilt nur dann, wenn $f_1 : f_2$, $\varphi_1 : \varphi_2$ konstant sind. Also haben wir den

Satz. *Wenn* $f_2(x)\,\varphi_2(x) > 0$ *und* $\varphi_1(x) : \varphi_2(x) \gtreqless B$ *(oder* $\lesseqgtr B$*) je nachdem*

$$f_1(x) : f_2(x) \gtreqless \int_a^b f_1\,\varphi_2\,dx : \int_a^b f_2\,\varphi_2\,dx$$

*ist, wo* $B$ *eine beliebige positive Zahl bedeutet, so ist*

$$\int_a^b f_1\,\varphi_1\,dx \int_a^b f_2\,\varphi_2\,dx \geqq (\text{oder} \leqq) \int_a^b f_1\,\varphi_2\,dx \int_a^b f_2\,\varphi_1\,dx,$$

*wo das Gleichheitszeichen gilt nur dann, wenn* $f_1 : f_2$, $\varphi_1 : \varphi_2$ *konstant sind.*

Setzt man $f_1 = f$, $\varphi_1 = \varphi$, $f_2 = \varphi_2 = 1$, so folgt der Brunnsche Satz.

Dass die Brunnsche Bedingung für die Gültigkeit der Ungleichung nicht notwendig ist, ist leicht ersichtlich. Wir können diese Bedingung wie folgt erweitern.

Damit der Ausdruck

$$\int_a^b f_1\,\varphi_1\,dx \int_a^b f_2\,\varphi_2\,dx - \int_a^b f_1\,\varphi_2\,dx \int_a^b f_2\,\varphi_1\,dx$$

$$= \int_a^b \int_a^b F(x, y) \left( \frac{f_1(x)}{f_2(x)} - A \right) \left( \frac{\varphi_1(x)}{\varphi_2(x)} - B \right) dx\,dy$$

$$- \int_a^b f_2(x)\,\varphi_2(x) \left( \frac{f_1(x)}{f_2(x)} - A \right) dx . \int_a^b f_2(y)\,\varphi_2(y) \left( \frac{\varphi_1(y)}{\varphi_2(y)} - B \right) dy$$

$\geqq 0$ sei, ist es schon hinreichend, dass

(1) $\quad f_2(x)\,\varphi_2(x) > 0$;

(2) $\quad \varphi_1(x) : \varphi_2(x) \gtrless B$, je nachdem $f_1(x) : f_2(x) \gtrless A$;

(3) $\quad \int_a^b f_2\,\varphi_2 \left( \frac{f_1}{f_2} - A \right) dx > 0$, (oder $< 0$),

$$\int_a^b f_2\,\varphi_2 \left( \frac{\varphi_1}{\varphi_2} - B \right) dx < 0, \text{ (oder } > 0);$$

d. h.

$$A < \int_a^b f_1\,\varphi_2\,dx : \int_a^b f_2\,\varphi_2\,dx, \text{ (oder } > 0),$$

$$B > \int_a^b f_2\,\varphi_1\,dx : \int_a^b f_2\,\varphi_2\,dx, \text{ (oder } < 0).$$

Darin bedarf man keiner Beschränkung auf $B$, wenn speziell

$$A=\int_a^b f_1\,\varphi_2\,dx : \int_a^b f_2\,\varphi_2\,dx$$

ist ([1]).

Sendai, September 1918.

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([1]) Herr Ogura hat mich freundlich darauf aufmerksam gemacht, dass die Ungleichung, welche ich in meiner vorigen Arbeit Herren Richardson—W. A. Hurwitz zuschrieb, schon von Herren Prof. I. Schur in Math. Ann. 67, 1909, angegeben wurde.

Die Herleitung der Schwarzschen Ungleichung aus der Integraldarstellung rührt soweit ich kenne, zuerst von Herrn Teixeira her (Darboux Bull. 22, 1885). In dieser Arbeit hat er auch von der Tchebycheffschen Ungleichung auseinandersetzt, aber es scheint dass er dieselbe Integraldarstellung als die gemeinsame Quelle beider Ungleichungen nicht bemerkt hatte.

Neulich habe ich gefunden, dass Stieltjes (Hermite-Stieltjes, Correspondance, t. 2, p. 191) auch eine Verallgemeinerung der Tchebycheffschen Ungleichung gegeben hatte. Seine Ungleichung ist der Zwischenfall von Tchebyscheffschen und meinigen Formeln, und zwar von der Form

$$\int f\,\varphi\,dx \int \psi\,dx \geqq \int \varphi\,dx \int f\,\psi\,dx.$$

Herr Rogers hat auch einige Ungleichungen von unserem Typus behandelt; insbesondere war die Formel (5) in meiner vorgen Arbeit Schon von ihm gefunden (Messenger of Math. (2) 17, 1888).

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## Berichtigung

zu meiner Note: Bemerkung zur Theorie der Approximation der irrationalen Zahlen durch rationale Zahlen, in diesem Journal, 14 (1918), pp. 109–115.

Seite 111, Z. v. u. 3: $a_{n+1} \geqq 2$ oder $a_{n+2} \geqq 2$,
anstatt $a_{n+1} < 2$ oder $a_{n+2} < 2$.

Seite 112, Z. v. u. 8: $\lambda+\sqrt{\lambda^2-4}$ anstatt $\sqrt{\lambda^2-4}$.

Nach der freundlichen Mitteilung des Herrn Prof. Takagi bleibt die Relation $a_{n+1} < \sqrt{\lambda^2-4}$ auch gültig, wenn $\lambda+\sqrt{\lambda^2-4}$ rational ist, was aus der Relation (8′) folgt.

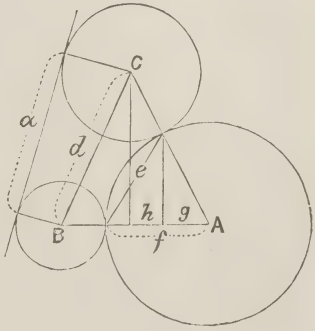
# 日本數學上ニ於ケル CASEY ノ定理.

三上義夫 (東京).

一ツノ圓ニ切スル四圓ノ六切線ノ長サニ關スル Casey ノ定理ハ和算家ノ遺書中ニモ多ク散見スル所ニシテ, Casey ヨリモ其年代ノ先テルハ注意スベキコトナリ. 林鶴一氏ハ東北數學雜誌第一卷 (1912) (p. 204-6) ニ此事ヲ略記セリ. Casey ノ Sequel to Euclid, 1900, p. 104 ニ依レバ, Casey ハ 1866 ニ得タルモノナレドモ (小倉氏譯さーもん解析幾何學, p. 192 ニハ 1857 トス), 和算家ハ少クモ文政十二三年即チ 1829-30 年頃ニハ之ヲ知リタリシナリ. 即チ英國ノ Casey ニ先ツコト三十年前ニアリ. 文政十三年ノ作ニ係レル寫本數理無盡藏ニ之ヲ載ス. 其事ニツキテハ余ハ東洋學藝雜誌第 30 卷 (1913) ニ和算家ノ幾何學諸定理ヲ擧ゲタル中ニ之ヲ示セリ. 此書ハ白石長忠閲, 門人池田貞一編ノ公式集ニシテ, 蓋シ白石長忠自身ノ手ニ成レルモノナリ. 和算書ハ某々閲トセルハ多クハ其人ノ著作ナリシコト一般ノ風習ナリキ. 尙白石長忠ガ文政十二年ニハ既ニ此定理ヲ使用セルコトハ後ニ之ヲ示スベシ.

數理無盡藏ハ公式ヲ擧ゲタルノミニシテ, 其證明ヲ說カズ. サレド和算家ガ此定理ニツキテ如何ナル證明ヲ用ヒタルカヲ明カニスルモ, 蓋シ無用ノ業ニアラザルベシ. 此證明ハ幾多ノ寫本ニ說ク所ナレドモ, 尾張ノ人吉田爲幸ノ著書ナル寫本變局算法ノ終リニ平圓通解ト題セル一節アリテ, 最モ完備セルモノナルガ故ニ, 今此書ニツキテ其要旨ヲ示スコトトス.

第一圖




先ヅ第一圖ニ於テ三ツノ圓 $A, B, C$ ノ半徑ヲ夫々 $a, b, c$ トシ, $A$, $C$ 二圓ノ切點及ビ中心 $C$ ヨリ $AB$ ヘ二ツノ垂線ヲ引キタル上ニテ圖中ニ示メセル如キ長サヲ取リ, $a$ ヲ $B$, $C$ 二圓ノ共通外切線ノ長サトスベシ. 然ルトキハ二ツノ直角三角形ノ相似ニヨリテ

$$g=fa/(a\pm c),$$

從テ $$h=a-g=a-fa/(a\pm c).$$

但シ二重符號ノ上者ハ $B, C$ 二圓ガ共ニ圓 $A$ ニ外切スル場合ニ關シ, 下者ハ二圓ガ共ニ內切スル場合ニ關スルモノナリ. 元來別々ニ記述シアレドモ, コヽニハ之ヲ合シテ述ブルコトトシタリ. 二圓ノ一ツガ $A$ ニ外切シ一ツハ內切スル場合ハ之ヲ論ゼズ. 上ノ式ノ兩邊ニ $2a$ ヲ乘ズルトキハ

$$c^2=h.\,2a=2a^2-2fa^2/(a\pm c).$$

然ルニ三角形ノ關係ニ依リテ

$$(a\pm b)^2+(a\pm c)^2-d^2=2(a\pm b)f.$$

此 $f$ ノ値ヲ上ノ式中ニ入ルレバ,

$$c^2=a^2\{2-[(a\pm b)^2+(a\pm c)^2-d^2]/(a\pm b)(a\pm c)\}$$

$$=a^2\{d^2-(b-c)^2\}/(a\pm b)(a\pm c).$$

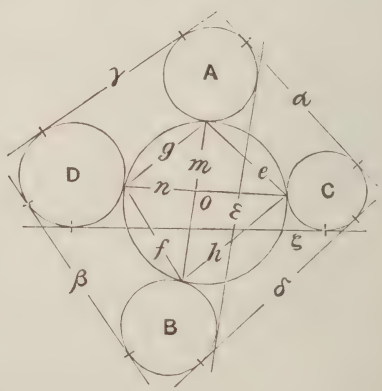
因テ $a$ ノ値ヲ計算シテ

$$c=a\,a/\sqrt{(a\pm b)(a\pm c)}.$$

此證明ハ池田貞一閔橋本昌方著算法點竄初學抄, 梅村重得ノ傍斜捷解, 岩田好算ノ圓內容八圓術解義等ニ見ル所ト大同少異ナリ. 中ニ就キテ點竄初學抄ハ刊本ニシテ, 其年代 (1830) モ亦最モ古シ.

上記ノ關係式ハ Casey ノ定理ヲ證スル爲メノ補助定理トナルモノナルガ, 諸種ノ命題ヲ解クニ甚ダ便利ナルモノニシテ, 和算家ハ好ンデ之ヲ應用シタリ. 數理無盡藏, 算法助術, 岡七兵衞ノ解路法等ノ公式集ニハ多ク之ヲ載ス. 特ニ梅村重得ノ傍斜捷解ニハ巧ニ之ヲ使用シテ多クノ問題ヲ解ケリ.

第二圖




次ニ第二圖ニ於テ四圓 $A, B, C, D$ ノ半徑ヲ $a, b, c, d$ トシ, 此四ツノ圓ハ共ニ一ツノ圓 $O$ ニ外切スルカ又ハ共ニ內切ストシ, 圓 $O$ ノ半徑ヲ $r$ トスベシ. 然ルトキハ圖ノ如ク長サヲ取レバ, 今求メタル補助關係ニヨリテ次ノ諸式ヲ得ベシ.

$$e=\frac{r\alpha}{\sqrt{(r\pm a)(r\pm d)}}, \qquad f=\frac{r\beta}{\sqrt{(r\pm b)(r\pm c)}},$$

$$g=\frac{r\gamma}{\sqrt{(r\pm a)(r\pm c)}}, \qquad h=\frac{r\delta}{\sqrt{(r\pm b)(r\pm d)}},$$

$$m=\frac{r\varepsilon}{\sqrt{(r\pm a)(r\pm b)}}, \qquad n=\frac{r\zeta}{\sqrt{(r\pm c)(r\pm d)}}.$$

今圓內六斜之適等又ハ圓內六斜術即チ圓ニ內接セル四邊形ノ四邊ト兩對角線トノ關係ニ依リテ

$$ef+gh=mn$$

ナリ. 此關係ハ即チ Ptolemy ノ定理ニ相當シ, 和算家ノ古クヨリ用ヒ來レルモノニシテ, 關流ノ傳書ナル括術ニモ既ニ現ハレ, 拾璣算法ニハ之ヲ公ニセリ. 其證明ハ圓內六斜之解, 圓內外三斜四斜適等, 圓內三斜術, 圓中三原適等, 天保十年 (1839) ニ野村貞處ノ訂補セル矩合樞要; 吉田茂一郎算術草稿, 川北朝鄰氏ノ圓內容八圓術等ニ說ク所最モ普通ノモノナリ. 吉田爲幸ノ遺稿ナル吉田先生解題, 岩田好算ノ圓內容八圓術解義ニハ稍々異ナレル證明ヲ用ヒ, 四圓四切題解義ニモ別ノ證明アリ. 恐クハ尙幾多ノ別證アルベシ.

前記ノ圓內六斜術ノ關係式ニ於テ既ニ得タル $e, f, \cdots$ ノ値ヲ入レ且ツ諸項ニ共通ノ因子ヲ省クトキハ,

$$\alpha\beta+\gamma\delta=\varepsilon\zeta$$

トナル. 是レ西洋數學ニ謂フ所ノ Casey ノ定理ニ外ナラズ. 其證明ハ代數的ナレドモ, 完全ニ成立セルヲ見ルベシ. 吉田氏ノ著書ニハ四圓ガ凡テ一ツノ圓ニ外切スルカ又ハ共ニ內切スル場合ヲ論ジ, 或ルモノハ內切シ或ルモノハ外切スル場合ヲ論ゼズ. 數理無盡藏ニ於テモ亦然リ.

和算書ニハ此定理ヲ載セタルモノ若クハ之ガ證明ヲ說ケルモノ少ナカラザレド, 內切ト外切トノ二ツノ場合ヲ共ニ記ルセルハ, 余ノ知ル限リニ於テハ獨リ上記ノ二書ト他ニ一ノ算法礎ナル刊本アルノミ.

數理無盡藏ニハ證明ヲ載セザルモ, 吉田氏ノ證明中ニ見エタル補助定理ヲモ載セ, 又內接四邊形ノ關係ヲモ擧グルヨリ見ルトキハ, 恐ク

ハ同ジ證明ヲ用ヒタルモノナルベク,此證明モ亦文政十三年ノ頃ヨリ行ハレタリト見テ大過ナカルベキヲ信ズ.

岩田好算ノ圓內容八圓術解義(乙丑五月二十二日改正,但シ乙丑ハ1865年),川北朝鄰氏ノ圓內容八圓術(文久二年 1862 脫稿,1866 訂書),著者未詳ノ寫本四圓四切題解義中ノ四切圓別記,梅村重得ノ傍斜捷解等ニモ前記ト同樣ニ四圓六切線ノ關係ヲ證明セリ. 梅村氏ノ著書ハ四圓ガ外切ノ場合ニ限リ,他ノ諸書ハ內切ノ場合ニ限レリ.

右ニツキテ四圓四切題解義ハ表紙ノ裏ニ松岡正之トアレバ或ハ此人ノ著書ナルベク,中ニ算法淺問抄等ノ問題ノ解義アルユヘ其年代モ大概明カナリ.

梅村重得ハ天保ノ頃長谷川氏ニ入門セル人ニシテ,其師ニ送レル書簡ニ依レバ氏ガ此定理ニ注意セルハ其以後ナルヲ知ルベシ.

前記ノ吉田爲幸モ御粥安本ノ門人ニシテ,其年代ハ梅村,松岡等ト略々同時ニアルベシ. 故ニ此等ノ證明ハ皆數理無盡藏ヨリ古キモノニアラザルヲ察スベシ. 川北朝鄰氏藏ナル御粥氏ノ自筆草稿中ニモ亦同ジ證明ヲ見ル. 四圓ノ內切セル場合ニツキテ說キタレドモ,吉田氏等ノ記述ハ蓋シコレヨリ傳ハレルモノナルベシ. 御粥氏ノ著書中ニ之ヲ見ルハ蓋シ注意ヲ要ス. 數理無盡藏ノ公式モ或ハ此人ヨリ出デタルヤモ未ダ知ルベカラザレバナリ.

溫知算叢ノ問題第四十三ハ,正方形ノ中ニ一ツノ圓ヲ畫キ,此圓ト各二邊トニ切スル四ツノ圓ヲ畫クトキハ,此四ツノ圓ノ直徑ヲ知リテ,正方形ノ一邊ノ長サヲ求ムルモノナリ.

算法雜俎ノ問題第四ハ,此正方形ノ代リニ矩形ヲ取リテ,一邊ヲ與ヘテ他ノ一邊ノ長サヲ問フモノナリ.

此二問題ハ四圓六切線ノ關係ヲ使用シテ解キ得ルコト論ヲ待タズ. 梅村重得ノ傍斜捷解上卷ノ如キモ現ニ此仕方ニテ之ヲ解キタル實例ヲ殘セリ. 林鶴一氏ガ初學算叢第二篇第三十四及ビ第三十九ノ兩卷ニ載セラレタリトセラルヽ梅村重得ノ解法(林氏前記論文)モ傍斜捷解ニ見ル所ト同一ナリ. 林氏ハ梅村重得ガ此種ノ解法ヲ用ヒタルノ故ニ,前記ノ溫知算叢,算法雜俎兩書ノ著者ハ果シテ如何ナル解法ヲ使用セルカ明カナラザレド,恐クハ既ニ Casey ノ定理ヲ使用セルモノナラントシ,因テ此定理成立ノ年代ヲ推定サレタリ. Gaston Bourgois 氏 (Les ancien-

nes mathématiques Japonaises, Paris, 1913, p. 30) ガ和算家ハ 1827 年(此年代恐クハ疑フベク, 或ハ溫知, 雜俎ヲ文政十年作トセルヨリ來レリ)ノ頃ニ之ヲ得タリシナラントセルハ蓋シ林氏ノ說ニ據レルモノナルベシ. サレド上記ノ二問題ハ別ノ仕方ニテモ亦解キ得ベキガ故ニ, 梅村重得ノ解法アルガ爲メニ直チニ其推定ヲ試ムルハ, 蓋シ非ナリ. 吉田爲幸ノ溫知算叢解ニハ上述ノ問題ニツキテ, 直徑 $a, b, c$ ナル三圓ガ $p$ ナル圓ニ切スルトキ, $b, c$; $c, a$; $a, b$ ノ共通切線ヲ順次ニ $\alpha, \beta, \gamma$ トスルトキハ,

$$4\alpha^2\beta^2\gamma^2+\Sigma(a+p)\alpha^4-2\Sigma\alpha^2\beta^2(a+p)(b+p)=0$$

ナル關係アリトシ, 此關係ニ基キテ其解法ヲ試ミタリ. 其寫本ニハ此關係ノ證明ハ此解別記ト記シテ之ヲ說カザレドモ, 梅村重得ノ傍斜捷解上卷及ビ御粥安本ノ草稿等ニハ前記ノ四圓六切線ノ證明ニ用ヒタル補助關係ニヨリテ之ヲ得タルモノニシテ, 四圓六切線ノ關係即チ Casey ノ定理ニハ何等ノ關係ナシ.

算法雜俎ノ問題モ亦同ジ關係ヲ使用シテ說キ得ルコト論ヲ俟タザルナリ.

長谷川西磻創術秋田義一編(實ハ長谷川ノ著書)ノ算法極形指南(天保六年 1835) ニハ此兩問題ヲ極形術ナル算法ニ依リテ解キ, 其解法ハ四圓六切線ノ關係ニ依レルモノトハ全ク異ナレリ. 白石長忠ガ文政二年 (1819) ニ作レル草稿ニモ四圓六切線ノ關係ヲ使用セズシテ, 所謂四圓傍斜術即チ二圓ノ切線ヲ此二圓ト相切シ且ツ互ニ相切スル他ノ二圓トノ關係上ヨリ說ケルモノニ依リテ解キタルモノアリト云フ.

事情斯ノ如クナレドモ, コヽニ一ノ注意スベキコトアリ. 川北朝鄰氏ノ數理起源卷四及ビ卷二十五ニ溫知算叢ノ問題ノ解ヲ載ス. 其解法ノ趣意ハ共ニ同一ナリ. 即チ相對スル二圓ノ共通切線ノ長サヲ二圓ノ中心距離ト直徑トニテ表ハシ, 其中心距離ニ代フルニ正方形ノ邊ト其直徑トノ函數ヲ以テシ, 因テ四圓六切線ノ關係中ニ於テ切線ノ長サニ代フルニ邊ト四圓ノ直徑トニテ表ハセルモノヲ以テシ, コレヨリ邊ノ長サヲ求メタルモノナリ. 梅村重得ノ著書ニ見ル所モ固ヨリ之ト同一ナリ.

算法雜俎第四問ノ解モ川北氏ノ著書第二十五卷ニ之ヲ說ケリ. 其解法ハ趣意同ジキガ故ニ別ニ言フマデモナシ.

川北氏ノ數理起源ハ幾多問題ノ解ヲ輯錄セルモノニシテ，其凡例ニハ

書中多ク余ガ解ク所ノモノナリト雖モ，間々亦鄰々白石先生ノ解ヲ潤色ス．社盟算譜，溫知算叢，算法雜俎ノ類是ナリ

ト見エ，又第二十五卷ノ卷首目錄ノ前ニハ

鄰々先生雜解集四十七番中ノ題又余自考少補シテ載之．

ト記ルシ，同卷中ナル前記溫知算叢問題ノ解ノ終リニ

文政十二年己丑三月六日考成

ト署セリ．サレバ此解義ハ白石長忠ノ手ニ成レルモノヲ川北氏ガ此集中ニ收メラレタルモノナルコト疑フベカラズ．

之ニ依リテ白石長忠ハ文政十二年 (1829) ニハ既ニ Casey ノ定理ヲ知リテ之ヲ適用シタリシヲ知ルナリ．

算法雜俎ハ岩井重遠ノ編ナレド，白石長忠ノ閲ニシテ，岩井ハ白石ノ門人ナリ．而シテ岩井氏ノ序文ニ依レバ

且今玆先生ノ高弟池田純夫子鄰々先生自得ノ妙旨……洩スコトナク輯錄シテ數理無盡藏ト云フ．……今其意ヲ用テ一二ノ題術ヲ撰ビ……先生ノ一閲ヲ請ヒ算法雜俎ト名ケ……

ト記シタリ．且ツ此書ハ其一部分ヲ除クノ外ハ白石氏ノ手ニ成リタルモノナリト推定スルニ足ルベキ書簡ノ存スルモノアリ．果シテ然ラバ，此書中ニ於ケル矩形內ニ四圓ヲ容レタル問題ノ解ハ上記ノ如キ解法ニ依リテ得タルモノナルベシ．此書ハ序文モ跋モ皆文政十三年 (1830) ニシテ，數理無盡藏ト同年ニ成リタリ．

溫知算叢ノ著者木村尙壽モ亦白石長忠門下ノ人ニシテ，此書亦白石氏ノ閲ニ係ル．然レドモ其序文ハ文政十一年 (1828) ト記ルセリ．即チ數理起源中ニ於ケル此問題ノ解トハ一年前ニ在リ．サレバ此問題ノ解ハ或ハ四圓六切線ノ關係ヲ使用セズシテ得タルモノナルヤモ亦知ルベカラズ．現ニ文政二年即チ十年前ニ於テ白石長忠ハ別ノ仕方ニテ此問題ヲ解セル草稿ヲ殘セリ．文政初年ニハ恐クハ白石長忠ハ未ダ四圓六切線ノ關係ヲ知ルニ至ラザリシニヤアルベシ．然レドモ要スルニ，此定理ハ文政十二年 (1829) 春ヨリ晩カラザル時ニ和算家ノ用ヒ始メタルモノナルコト疑フベカラズ．

以上說ク所ニ依リテ考フルニ，和算家ガ四圓六切線ノ關係即チ

Casey ノ定理ヲ案出シテ使用スルニ至レルハ, 蓋シ文政末年ノコトナルベク, 其以前ニハ未ダ成立セザリシモノナルベシ. 而シテ此定理ノ成立ニツキテハ, 上記ノ和算家諸氏ガ其證明ニ使用シタル補助定理ニ重大ナル關係アリシ如ク思ハル. 此補助定理ノ創意サレタルガ爲メニ其關係上ヨリ四圓六切線ノ定理ニモ達スルコトヲ得タルモノナルベシ. 此補助定理モ亦文政頃ヨリ始メテ使用サルヽコトトナレルモノナラン. 年代ノ少シク先テルモノニアリテハ安島直圓ノ如キ大家ノ諸著述中ニモ未ダ之ヲ見出スコト能ハズ. 諸算書ノ之ヲ載セタルモノニハ算法點竄初學抄, 數理無盡藏, 解路法, 矩合輯略, 算法助術, 算法雜解, 矩合樞要, 御粥安本ノ草稿, 梅村重得ノ著書等アリ. 今此等ノ諸算書ヲ見ルニ, 文政十三年ノモノヲ最古トシ, 年代未詳ノ諸書モ皆此以前ニ屬スル如ク思ハルヽモノナシ. サレバ此補助定理ハ四圓六切線ノ定理ト同ジ頃ニ現ハレ來レルヲ見ルベシ. 或ハ此以前ニ於ケル未知ノ算書ニ之ヲ說ケルモノアラン. 然モ當時ノ諸算書ニ散見セザルヨリ推ストキハ, 未ダ多ク使用サレザリシヲ想フベキナリ. 此補助定理ノ成リテヨリハ, 安島直圓以來好ンデ用ヒラレタル四圓傍斜術ハ漸ク跡ヲ潛メ, 全ク之ニ代ルニ至レリ.

之ニ依リテ考フルトキハ, 此補助定理成リテ諸圓ノ切線ニ關スル問題等ヲ論ズル上ニ大ナル便利ヲ生ズルコトトナリ, 多ク此種ノ問題ヲ取扱ヘルヨリ端ナクモ四圓六切線ノ定理即チ Casey ノ定理ニ逢着スルニ至レルモノナルベシ.

和算家ガ四圓六切線ノ定理ヲ證明スルニハ何レモ上記ノ補助定理ニ依賴セルモノニシテ, 別ノ證明ヲ試ミタルハ獨リ大村一秀ノ遺稿中ニ現ハレタルノミナリト云フ. 兩者ノ密接ナル關係アリシコト勿論タルナリ.

尙此問題ニツキテ一ノ注意スベキコトアリ. 以上說ク所ハ皆寫本ニ現ハレタルモノニ過ギザレドモ, 刊本ニモ之ヲ載セタルモノアルコト是レナリ. 是ノ刊本ハ即チ算法礎ナリ. 文久二年 (1862) 喜多川孟孰(或ハ孟敦)ガ金澤ニテ刊行セルモノナリ. 而シテ更ニ明治六年 (1873) ニ至リ加賀國松原一記著ト改メテ印行セルモノアリ. 固ヨリ序文末ノ姓名ト年號トヲ變更シタルニ過ギズ. 喜多川氏ハ加州大野ノ人ニシテ後ニ松原姓ヲ冒セリ.

此書ニ四圓六切線ノ定理ハ 133 番及ビ 134 番ノ問題トシテ其公式ノミ之ヲ擧ゲタリ. 前者ハ四圓ガ皆一圓ニ外切スル場合ニシテ, 後者ハ皆一圓ニ內切スル場合ニ關ス.

此書ノ附錄ニハ算法雜俎ノ問題及ビ他ノ一問題ニ此定理ヲ應用シテ解ケル解義ヲ與フ.

此刊行ノ算書算法礎モ文久二年ノ版ニ於テハ Casey ノ公ニシタル 1866 ニ先ツコト實ニ四年前ニアリ.

# On Mr. Ôue's Theorem,

by

KYÔICHI ÔISHI, Sendai.

Extending Messrs. Ôue and Agronomof's theorems ([1]), I will here give the following more general theorem including them as special cases: *Let $m$ points be given on a curve $C$ which is determined by $n$ points on it, $m$ being greater than or equal to $n$; then centroids of every $m-1$ points taken out of these $m$ points lie on a curve $C'$ similar and similarly situated to $C$. Let the curve $C$ be central; and take $m+1$ points on it and determine $m+1$ curves $C'$ for every $m$ point-set, chosen from these $m+1$ points; then these $C'$ curves meet in a point $P'$. Take $m+2$ points on $C$ and find $m+2$ points $P'$ for every $m+1$ point-set; then these points lie on a curve $C''$ similar and similarly situated to $C$; and so on.*

Let $z$ be the complex coordinate of the variable point on the original curve, and those of the given points be $z_1, z_2, \cdots\cdots, z_m$. Then the centroids of $m-1$ points of these are given by.

$$\left.\begin{array}{l} \dfrac{1}{m-1}(z_2+z_3+\cdots\cdots+z_m), \\ \dfrac{1}{m-1}(z_1+z_3+\cdots\cdots+z_m), \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ \dfrac{1}{m-1}(z_1+z_2+\cdots\cdots+z_{m-1}). \end{array}\right\} \tag{1}$$

Now applying the linear transformation

$$\bar{z}=\frac{1}{m-1}(z_1+z_2+\cdots\cdots+z_m)-\frac{1}{m-1}z, \tag{2}$$

we get a curve $\bar{z}$ which is similar and similarly situated to the original curve $z$. It is easy to see that the points expressed by (1) lie on the curve $\bar{z}$.

When the original curve is a central curve, we can further say that: if we take one point more whose coordinate is $z_{m+1}$ on the curve, and

([1]) This Journal, Vol. 10 (1916), p. 225.

describe the curves $z$'s from every $m$ points, then these curves will be concurrent.

For, the curves are given by

$$\left.\begin{array}{l} \dfrac{1}{m-1}(z_2+z_3+\cdots\cdots+z_{m+1})-\dfrac{1}{m-1}z, \\ \dfrac{1}{m-1}(z_1+z_3+\cdots\cdots+z_{m+1})-\dfrac{1}{m-1}z, \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ \dfrac{1}{m-1}(z_1+z_2+\cdots\cdots+z_m)-\dfrac{1}{m-1}z. \end{array}\right\} \tag{3}$$

We may take, without loss of generality, the origin of coordinates at the centre of the original curve, so that the points $-z_1, -z_2, \cdots\cdots, -z_{m+1}$ also lie on it. By putting these in the expressions (3) respectively, it can be seen that the curves (3) pass through the same point

$$\frac{1}{m-1}(z_1+z_2+\cdots\cdots+z_{m+1}). \tag{4}$$

If we take one point more on the curve $z$, we see that the points, such as (4), for every $m+1$ points lie on a curve similar and similarly situated to the curve $z$; and so on. The theorem is thus proved.

In addition to this, since the centres of the curves (3) are

$$\left.\begin{array}{l} \dfrac{1}{m-1}(z_2+z_3+\cdots\cdots+z_{m+1}), \\ \dfrac{1}{m-1}(z_1+z_3+\cdots\cdots+z_{m+1}), \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ \dfrac{1}{m-1}(z_1+z_2+\cdots\cdots+z_m), \end{array}\right\} \tag{5}$$

we see that the centres of the curves (3) lie on the curve

$$\frac{1}{m+1}(z_1+z_2+\cdots\cdots+z_{m+1})-\frac{1}{m-1}z, \tag{6}$$

which is similar and similarly situated to the curve $z$; and further taking one point more $z_{m+2}$ on the curve $z$, we see that such curves as (6), viz.

$$\frac{1}{m-1}(z_2+z_3+\cdots\cdots+z_{m+2})-\frac{1}{m-1}z, \Big)$$

$$\left.\begin{array}{l} \dfrac{1}{m-1}(z_1+z_3+\cdots\cdots+z_{m+2})-\dfrac{1}{m-1}z, \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ \dfrac{1}{m-1}(z_1+z_2+\cdots\cdots+z_{m+1})-\dfrac{1}{m-1}z, \end{array}\right\} \tag{7}$$

meet in the point

$$\frac{1}{m-1}(z_1+z_2+\cdots\cdots+z_{m+2}).$$

and at the same time, the centres of the curves (7), viz.,

$$\left.\begin{array}{l} \dfrac{1}{m-1}(z_2+z_3+\cdots\cdots+z_{m+2}), \\ \dfrac{1}{m-1}(z_1+z_3+\cdots\cdots+z_{m+2}), \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ \dfrac{1}{m-1}(z_1+z_2+\cdots\cdots+z_{m+1}) \end{array}\right\}$$

lie on the curve

$$\frac{1}{m-1}(z_1+z_2+\cdots\cdots+_{m+2})-\frac{1}{m-1}z,$$

which is similar and similarly situated to the original curve $z$; and so on.

If the original curve be the circle, we get the theorem of Mr. N. Agronomof(1) and thus when $m=3$ we have the theorem of Messrs. Ôue and Coolidge(2).

(1) This Journal, vol. 11 (1917), p. 243, and *Extension d'un théorèm de M. S. Oüe* Nouvelles Annales de Math, 4. Ser., t. 18 (1918), p. 256.

(2) Coolidge, *Some circles associated with concyclic points*, Annals of Mathematics, 2nd Ser., vol. 12, 1910-1911, p. 39.

# Some Theorems on Limits,

by

SEIMATSU NARUMI, Sendai.

Let us consider the sequence of real or complex numbers $x_1, x_2, x_3, \cdots\cdots, x_n, \cdots\cdots$, and put

$$S_n^{(0)}(x)=x_n, \quad S_n^{(1)}(x)=\left\{x_1+x_2+\cdots\cdots+x_n\right\}/n,$$

$$S_n^{(2)}(x)=\left\{S_1^{(1)}(x)+S_2^{(1)}(x)+\cdots\cdots+S_n^{(1)}(x)\right\}/n,$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$S_n^{(k)}(x)=\left\{S_1^{(k-1)}(x)+S_2^{(k-1)}(x)+\cdots\cdots+S_n^{(k-1)}(x)\right\}/n.$$

The well known theorem on limit due to Cauchy ([1]) may be stated as follows:

Theorem A. If the sequence $x_n$ converges to the limit $l$, then the sequences $S_n^{(1)}(x)$, $S_n^{(2)}(x)$, $\cdots\cdots$, $S_n^{(k)}(x)$ converge to the same limit $l$.

Thus we obtain the equation

$$\lim_{n=\infty}\left\{a_0 S_n^{(0)}(x)+a_1 S_n^{(1)}(x)+a_2 S_n^{(2)}(x)+\cdots\cdots+a_k S_n^{(k)}(x)\right\}$$
$$=l\,(a_0+a_1+a_2+\cdots\cdots+a_k). \tag{1}$$

But the converse of this theorem is in general untrue; in other words, from the existence of the limit (1) we can not deduce the convergency of the sequence $x_n$.

In my previous note ([2]) I have proved the following theorem:

If the real parts of the first $h$ roots, for instance, of the equation

$$\lambda^k-a_1\lambda^{k-1}+a_2\lambda^{k-2}-a_3\lambda^{k-3}+\cdots\cdots+(-1)^k a_k=0$$

are less than $-1$, while the others are greater than $-1$, then the relation

$$\lim_{n=\infty}\left\{S_n^{(0)}(x)+a_1 S_n^{(1)}(x)+a_2 S_n^{(2)}(x)+\cdots\cdots+a_k S_n^{(k)}(x)\right\}=l$$

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([1]) Cours d'Analyse de l'École royale polytechnique (1821); Œuvres complétes, 2me série, t, 3., pp. 54–58.

([2]) S. Narumi, A theorem on limits, this Journal, vol. 12 (1917), pp. 275–290.

implies the relation

$$\lim_{n=\infty} x_n = l \Big/ (1 + a_1 + a_2 + \cdots\cdots + a_k),$$

when and only when

$$\lim_{n=\infty} n^{1+\lambda_p} S_n^{(1)}(x^{(p)}) = 0$$

for all values of $p = 1, 2, 3, \cdots\cdots, h$, where

$$x_n^{(p)} + \lambda_p S_n^{(1)}(x^{(p)}) = x_n^{(1+p)},$$

$$S_n^{(1)}(x^{(p)}) = \left\{ x_1^{(p)} + x_2^{(p)} + x_3^{(p)} + \cdots\cdots + x_n^{(p)} \right\} / n$$

and $$\lambda^k + a_1 \lambda^{k-1} + a_2 \lambda^{k-2} + \cdots\cdots + a_k = (\lambda + \lambda_1)(\lambda + \lambda_2) \cdots\cdots (\lambda + \lambda_k).$$

A particular case of this theorem may be stated as follows:

Theorem B. When the sequence

$$a_0 S_n^{(0)}(x) + a_1 S_n^{(1)}(x) + a_2 S_n^{(2)}(x) + \cdots\cdots + a_k S_n^{(k)}(x)$$

has a finite limit $l(a_0 + a_1 + a_2 + \cdots\cdots + a_k)$, the necessary and sufficient condition that the sequence $x_n$ should converge to the limit $l$ is that the real parts of all the roots of the equation

$$a_0 \lambda^k + a_1 \lambda^{k-1} + a_2 \lambda^{k-2} + \cdots\cdots + a_k = 0$$

should be less than 1.

## I.

At first, I will generalize the last theorem in the following manner: Let the sequence $x_1, x_2, x_3, \cdots\cdots, x_n, \cdots\cdots$ be of such a nature that

$$\left.\begin{aligned}
&\lim_{n=\infty} S_n^{(0)}(\xi_1) \\
&\qquad = \lim_{n=\infty} \left\{ a_{1,0} S_n^{(0)}(x) + a_{1,1} S_n^{(1)}(x) + \cdots\cdots + a_{1,k_1} S_n^{(k_1)}(x) \right\} \\
&\qquad = l(a_{1,0} + a_{1,1} + \cdots\cdots + a_{1,k_1}), \\
&\lim_{n=\infty} S_n^{(0)}(\xi_2) \\
&\qquad = \lim_{n=\infty} \left\{ a_{2,0} S_n^{(0)}(x) + a_{2,1} S_n^{(1)}(x) + \cdots\cdots + a_{2,k_2} S_n^{(k_2)}(x) \right\} \\
&\qquad = l(a_{2,0} + a_{2,1} + \cdots\cdots + a_{2,k_2}), \\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
&\lim_{n=\infty} S_n^{(0)}(\xi_i)
\end{aligned}\right\} \qquad (2)$$

$$\left.\begin{aligned}&=\lim_{n=\infty}\left\{a_{i,0}\,S_n^{(0)}(x)+a_{i,1}\,S_n^{(1)}(x)+\cdots\cdots+a_{i,k_i}\,S_n^{k_i}(x)\right\}\\&=l\,(a_{i,0}+a_{i,1}+\cdots\cdots+a_{i,k_i}),\end{aligned}\right\}$$

then the necessary and sufficient condition that we can ensure the relation

$$\begin{aligned}&\lim_{n=\infty}\left\{b_0\,S_n^{(0)}(x)+b_1\,S_n^{(1)}(x)+\cdots\cdots+b_h\,S_n^{(h)}(x)\right\}\\&\qquad=l\,(b_0+b_1+\cdots\cdots+b_h)\end{aligned}\tag{3}$$

is that there is no common root between $f_1(\lambda)=0, f_2(\lambda)=0, \cdots\cdots, f_i(\lambda)=0$, which does not make $g(\lambda)$ vanish and whose real part is not less than 1, where

$$\begin{aligned}f_1(\lambda)&=a_{1,0}\,\lambda^{k_1}+a_{1,1}\,\lambda^{k_1-1}+\cdots\cdots+a_{1,k_1},\\f_2(\lambda)&=a_{2,0}\,\lambda^{k_2}+a_{2,1}\,\lambda^{k_2-1}+\cdots\cdots+a_{2,k_2},\\&\cdots\cdots\cdots\cdots\cdots\cdots\\f_i(\lambda)&=a_{i,0}\,\lambda^{k_i}+a_{i,1}\,\lambda^{k_i-1}+\cdots\cdots+a_{i,k_i},\\g(\lambda)&=b_0\,\lambda^h+b_1\,\lambda^{h-1}+\cdots\cdots+b_h.\end{aligned}$$

To prove this theorem, let $\varphi(\lambda)$ be the greatest common divisor of $f_1(\lambda), f_2(\lambda), \cdots\cdots f_i(\lambda)$. In particular, if $f_1(\lambda), f_2(\lambda), \cdots\cdots, f_i(\lambda)$ are relatively prime to one another, then $\varphi(\lambda)$ reduces to a constant, and in case $i=1$, $\varphi(\lambda)$ is identical with $f_1(\lambda)$; comprising these particular cases, $\varphi(\lambda)$ may be expressed in the following form

$$\varphi(\lambda)=c_0\,\lambda^p+c_1\,\lambda^{p-1}+\cdots\cdots+c_p.$$

In Algebra we know that a series of polynomials $F_1(\lambda), F_2(\lambda), \cdots\cdots F_i(\lambda)$ can be found, so that

$$f_1(\lambda)\,F_1(\lambda)+f_2(\lambda)\,F_2(\lambda)+\cdots\cdots+f_i(\lambda)\,F_i(\lambda)=\varphi(\lambda)$$

is identically satisfied, where

$$\begin{aligned}F_1(\lambda)&=a_{1,0}\,\lambda^{k_1'}+a_{1,1}\,\lambda^{k_1'-1}+\cdots\cdots+a_{1,k_1'},\\F_2(\lambda)&=a_{2,0}\,\lambda^{k_2'}+a_{2,1}\,\lambda^{k_2'-1}+\cdots\cdots+a_{2,k_2'},\\&\cdots\cdots\cdots\cdots\cdots\cdots\\F_i(\lambda)&=a_{i,0}\,\lambda^{k_i'}+a_{i,1}\,\lambda^{k_i'-1}+\cdots\cdots+a_{i,k_i'},\end{aligned}$$

and $$k_1+k_1'=k_2+k_2'=\cdots\cdots=k_i+k_i'=m.$$

Putting

$$f_1(\lambda)\,F_1(\lambda)=A_{1,0}\,\lambda^m+A_{1,1}\,\lambda^{m-1}+\cdots\cdots+A_{1,m},$$
$$f_2(\lambda)\,F_2(\lambda)=A_{2,0}\,\lambda^m+A_{2,1}\,\lambda^{m-1}+\cdots\cdots+A_{2,m},$$
$$\cdots\cdots\cdots\cdots\cdots\cdots$$
$$f_i(\lambda)\,F_i(\lambda)=A_{i,0}\,\lambda^m+A_{i,1}\,\lambda^{m-1}+\cdots\cdots+A_{i,m},$$

and applying Theorem A, we obtain

$$\lim_{n=\infty}\left\{a_{1,0}\,S_n^{(1)}(\xi_1)+a_{1,1}\,S_n^{(1)}(\xi_1)+\cdots\cdots+a_{1,\,k_1'}\,S_n^{(k_1')}(\xi_1)\right\}$$
$$=l\,(a_{1,0}+a_{1,1}+\cdots\cdots+a_{1,\,k_1})\,(a_{1,0}+a_{1,1}+\cdots\cdots+a_{1,\,k_1'}),$$
$$\lim_{n=\infty}\left\{a_{2,0}\,S_n^{(0)}(\xi_2)+a_{2,1}\,S_n^{(1)}(\xi_2)+\cdots\cdots+a_{1,\,k_2'}\,S_n^{(k_2')}(\xi_2)\right\}$$
$$=l\,(a_{2,0}+a_{2,1}+\cdots\cdots+a_{2,\,k_2})\,(a_{2,0}+a_{2,1}+\cdots\cdots+a_{2,k_1'}),$$
$$\cdots\cdots\cdots\cdots\cdots\cdots$$
$$\lim_{n=\infty}\left\{a_{i,0}\,S_n^{(0)}(\xi_i)+a_{i,1}\,S_n^{(1)}(\xi_i)+\cdots\cdots+a_{i,\,ki'}\,S_n^{(k_i')}(\xi_i)\right\}$$
$$=l\,(a_{i,0}+a_{i,1}+\cdots\cdots+a_{i,k_i})\,(a_{i,0}+a_{i,1}+\cdots\cdots+a_{i,ki'}),$$

that is to say,

$$\lim_{n=\infty}\left\{A_{1,0}\,S_n^{(0)}(x)+A_{1,1}\,S_n^{(1)}(x)+\cdots\cdots+A_{1,m}\,S_n^{(m)}(x)\right\}$$
$$=l\,(A_{1,0}+A_{1,1}+\cdots\cdots+A_{1,m})=l\,f_1\,(1)\,F_1\,(1),$$
$$\lim_{n=\infty}\left\{A_{2,0}\,S_n^{(0)}(x)+A_{2,1}\,S_n^{(1)}(x)+\cdots\cdots+A_{2,m}\,S_n^{(m)}(x)\right\}$$
$$=l\,(A_{2,0}+A_{2,1}+\cdots\cdots+A_{2,m})=l\,f_2\,(1)\,F_2\,(1),$$
$$\cdots\cdots\cdots\cdots\cdots\cdots$$
$$\lim_{n=\infty}\left\{A_{i,0}\,S_n^{(0)}(x)+A_{i,1}\,S_n^{(1)}(x)+\cdots\cdots+A_{i,m}\,S_n^{(m)}(x)\right\}$$
$$=l\,(A_{i,0}+A_{i,1}+\cdots\cdots+A_{i,m})=l\,f_i\,(1)\,F_i\,(1).$$

By adding the above relations side by side, and using the relation

$$f_1(\lambda)\,F_1(\lambda)+f_2(\lambda)\,F_2(\lambda)+\cdots\cdots+f_i(\lambda)\,F_i(\lambda)=\varphi(\lambda)$$

we have

$$\lim_{n=\infty}\left\{c_0\,S_n^{(0)}(x)+c_1\,S_n^{(1)}(x)+\cdots\cdots+c_p\,S_n^{(p)}(x)\right\}$$
$$=l\,(c_0+c_1+\cdots\cdots+c_p). \qquad (3)$$

Let us consider, in the first place, the case in which the real parts of all the roots of the equation

$$\varphi(\lambda)=c_0\,\lambda^p+c_1\,\lambda^{p-1}+c_2\,\lambda^{p-2}+\cdots\cdots+c_p=0$$

are less than 1. Then by Theorem B, the relation (3) implies the relation

$$\lim_{n=\infty} x_n=l,$$

and therefore, by Theorem A, we have

$$\lim_{n=\infty}\left\{b_0\,S_n^{(0)}(x)+b_1\,S_n^{(1)}(x)+\cdots\cdots+b_h\,S_n^{(h)}(x)\right\}$$
$$=l\,(b_0+b_1+\cdots\cdots+b_h).$$

In the second place, suppose that the real parts of some roots of the equation $\varphi(\lambda)=0$ are not less than 1, while such roots are also the roots of the equation

$$g(\lambda)=b_0\,\lambda^h+b_1\,\lambda^{h-1}+\cdots\cdots+b_h=0.$$

If such roots be the roots of the equation

$$\psi(\lambda)=d_0\,\lambda^q+d_1\,\lambda^{q-1}+\cdots\cdots+d_q=0,$$

then, since the real parts of the other roots of $\varphi(\lambda)=0$ are less than 1, we see by Theorem B as before that the relation

$$\lim_{n=\infty}\left\{c_0\,S_n^{(0)}(x)+c_1\,S_n^{(1)}(x)+\cdots\cdots+c_p\,S_n^{(p)}(x)\right\}$$
$$=l\,(c_0+c_1+\cdots\cdots+c_p)$$

implies the relation

$$\lim_{n=\infty}\left\{d_0\,S_n^{(0)}(x)+d_1\,S_n^{(1)}(x)+\cdots\cdots+d_q\,S_n^{(q)}(x)\right\}$$
$$=l\,(d_0+d_1+d_2+\cdots\cdots+d_q).$$

Thus, since $\psi(\lambda)$ is also a factor of $g(\lambda)$, it follows by Theorem A that

$$\lim_{n=\infty}\left\{b_0\,S_n^{(0)}(x)+b_1\,S_n^{(1)}(x)+\cdots\cdots+b_h\,S_n^{(h)}(x)\right\}$$
$$=l\,(b_0+b_1+b_2+\cdots\cdots+b_h).$$

Finally consider the case in which there is, at least, one root of $\varphi(\lambda)=0$ which is not a root of $g(\lambda)=0$, and whose real part is not less than 1. In this case we can define $x_n$ such that although the limits (2) exist, the limit (3) does not exist. It is not difficult to construct ex-

amples to show the truth of this statement. The following is one of such examples.

Now let us suppose that $\lambda+a$ is a factor of $\varphi(\lambda)$, but not that of $g(\lambda)$, and that

$$-a=\alpha+i\beta,$$

where $\alpha \geqq 1$; then let $m$ be a positive integer such that

$$\alpha+1 \geqq m > \alpha.$$

Let us put

$$x_n = l - a\,K_n,$$

where
$$K_n = 0, \qquad \text{if } 1 \leqq n < m,$$
$$= \left\{ n \prod_{t=m}^{t=n} \left(1+\frac{a}{t}\right) \right\}^{-1}, \qquad \text{if} \quad n \geqq m.$$

This sequence has the required nature. For, we have

$$S_n^{(1)}(x) = l - \frac{a}{n} \sum_{p=1}^{p=n} K_p$$

and since ([1])

$$\sum_{p=1}^{p=n} K_p = \sum_{p=m}^{p=n} \frac{a}{p} \left\{ \prod_{t=m}^{t=p} \left(1+\frac{a}{t}\right) \right\}^{-1}$$
$$= \sum_{p=m}^{p=n} \left(1+\frac{a}{p}\right) \left\{ \prod_{t=m}^{t=p} \left\{1+\frac{a}{t}\right) \right\}^{-1} - \sum_{p=m}^{p=n} \left\{ \prod_{t=m}^{t=p} \left(1+\frac{a}{t}\right) \right\}^{-1}$$
$$= 1 + \sum_{p=m}^{p=n-1} \left\{ \prod_{t=m}^{t=p} \left(1+\frac{a}{t}\right) \right\}^{-1} - \sum_{p=m}^{p=n} \left\{ \prod_{t=m}^{t=p} \left(1+\frac{a}{t}\right) \right\}^{-1}$$
$$= 1 - \sum_{t=m}^{t=n} \left(1+\frac{a}{t}\right) \Big\}^{-1} = 1 - n\,K_n,$$

namely,

$$\sum_{p=1}^{p=n} K_p = \frac{1}{a} - \frac{n}{a} K_n.$$

It follows that

$$S_n^{(1)}(x) = l - \frac{1}{n} + K_n.$$

---

([1]) Hereafter we shall assume that $n \geqq m$.

Thus we have

$$x_n + a\, S_n^{(1)}(x) = l\,(1+a) - \frac{a}{n},$$

and therefore

$$\lim_{n=\infty} (x_n + a\, S_n^{(1)}(x)) = l\,(1+a).$$

Since the factor $\lambda + a$ is common to $f_1(\lambda), f_2(\lambda), \cdots\cdots, f_i(\lambda)$, by applying Theorem A, we obtain

$$\lim_{n=\infty}\left\{a_{1,0}\, S_n^{(0)}(x) + a_{1,1}\, S_n^{(1)}(x) + \cdots\cdots + a_{1,\,k_1}\, S_n^{(k_1)}(x)\right\}$$
$$= l\,(a_{1,0} + a_{1,1} + \cdots\cdots + a_{1,\,k_1}),$$
$$\lim_{n=\infty}\left\{a_{2,0}\, S_n^{(0)}(x) + a_{2,1}\, S_n^{(1)}(x) + \cdots\cdots + a_{2,\,k_2}\, S_n^{(k_2)}(x)\right\}$$
$$= l\,(a_{2,0} + a_{2,1} + \cdots\cdots + a_{2,\,k_2}),$$
$$\cdots\cdots\cdots\cdots\cdots\cdots$$
$$\lim_{n=\infty}\left\{a_{i,0}\, S_n^{(1)}(x) + a_{i,1}\, S_n^{(1)}(x) + \cdots\cdots + a_{i,k_i}\, S_n^{(k_i)}(x)\right\}$$
$$= l\,(a_{i,0} + a_{i,1} + \cdots\cdots + a_{i,k_i}).$$

On the other hand, employing the relations

$$x_n = S_n^{(0)}(x) = l - a\, K_n,$$
$$\frac{1}{n}\sum_{p=1}^{p=n} K_p = \frac{1}{-a}\left\{K_n - \frac{1}{n}\right\},$$

we can easily obtain

$$S_n^{(1)}(x) = l + (K_n - s_n^{(0)}),$$
$$S_n^{(2)}(x) = l + \frac{1}{-a}(K_n - s_n^{(0)} + a\, s_n^{(1)}),$$
$$S_n^{(3)}(x) = l + \frac{1}{(-a)^2}(K_n - s_n^{(0)} + a\, s_n^{(1)} - a^2\, s_n^{(2)}),$$
$$\cdots\cdots\cdots\cdots\cdots\cdots$$
$$S_n^{(h)}(x) = l + \frac{1}{(-a)^{h-1}}(K_n - s_n^{(0)} + a\, s_n^{(1)} - a^2\, s_n^{(2)} + \cdots\cdots - (-a)^{h-1}\, s^{(h-1)})$$

where

$$s_n^{(0)} = 0 \qquad \text{if} \qquad 1 \leqq n < m,$$

$$=\frac{1}{n} \qquad \text{if} \quad n \geqq m,$$

and
$$s_n^{(1)}=\frac{1}{n}(s_1^{(0)}+s_2^{(0)}+s_3^{(0)}+\cdots\cdots+s_n^{(0)}),$$

$$s_n^{(2)}=\frac{1}{n}(s_1^{(1)}+s_2^{(1)}+s_3^{(1)}+\cdots\cdots+s_n^{(1)}),$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$s_n^{(h-1)}=\frac{1}{n}(s_1^{(h-2)}+s_2^{(h-2)}+s_3^{(h-2)}+\cdots\cdots+s_n^{(h-2)}),$$

and therefore it follows that

$$\begin{aligned}
&b_0\,S_n^{(0)}(x)+b_1\,S_n^{(1)}(x)+b_2\,S_n^{(2)}(x)+\cdots\cdots+b_h\,S_n^{(h)}(x)\\
&=l\,(b_0+b_1+b_2+\cdots\cdots+b_h)\\
&+\frac{K_n}{(-a)^{h-1}}\left\{b_0\,(-a)^h+b_1\,(-a)^{h-1}+b_2\,(-a)^{h-2}+\cdots\cdots+b_h\right\}\\
&-\frac{1}{(-a)^{h-1}}\Bigg[\left\{b_1\,(-a)^{h-1}+b_2\,(-a)^{h-2}+\cdots\cdots+b_h\right\}s_n^{(0)}\\
&-a\left\{b_2\,(-a)^{h-2}+\cdots+b_h\right\}s_n^{(1)}+a^2\left\{b_3\,(-a)^{h-3}+\cdots+b_h\right\}s_n^{(2)}+\cdots\cdots\\
&\qquad\qquad+(-a)^{h-1}\,b_h\;s_n^{(h-1)}\Bigg]\\
&=l\,g(1)+\frac{1}{(-a)^{h-1}}\,g\,(-a)\,K_n\\
&\qquad-\frac{1}{(-a)^{h-1}}\sum_{p=1}^{p=h}s_n^{(p-1)}\sum_{i=p}^{i=h}b_i\,(-a)^{h-i},
\end{aligned}$$

that is to say,

$$\begin{aligned}
&\left\{b_0\,S_n^{(0)}(x)+b_1\,S_n^{(1)}(x)+\cdots\cdots+b_h\,S_n^{(h)}(x)\right\}-l\,(b_0+b_1+\cdots\cdots+b_h)\\
&=-\frac{1}{(-a)^{h-1}}\left\{\sum_{p=1}^{p=h}s_n^{(p-1)}\sum_{i=p}^{i=h}b_i\,(-a)^{h-i}-g\,(-a)\,K_n\right\}.
\end{aligned}$$

It is evident that

$$\lim_{n=\infty}\sum_{p=1}^{p=h}s_n^{(p-1)}\sum_{i=p}^{i=h}b_i\,(-a)^{h-i}=0,$$

since $s_n^{(0)}$ tends to zero when $n$ increases indefinitely, and at the same time $s^{(0)}, s_n^{(2)}, \cdots\cdots, s_n^{(h-1)}$ tend to zero, by Theorem A.

Next we observe that $g(-a)$ does not vanish and that the sequence $K_n$ does not tend to zero as $n$ tends to infinity. Thus from these results we can deduce that the sequence

$$b_0 S_n^{(0)}(x)+b_1 S_n^{(1)}(x)+b_2 S_n^{(2)}(x)+\cdots\cdots+b_h S_n^{(h)}(x)$$

does not converge to the limit $l(b_0+b_1+b_2+\cdots\cdots+b_h)$ as $n$ increases indefinitely.

We thus obtain the following theorem:

*When*

$$\lim_{n=\infty}\left\{a_{1,0} S_n^{(0)}(x)+a_{1,1} S_n^{(1)}(x)+a_{1,2} S_n^{(2)}(x)+\cdots\cdots+a_{1,k_1} S_n^{(k_1)}(x)\right\}$$
$$=l(a_{1,0}+a_{1,1}+a_{1,2}+\cdots\cdots+a_{1,k_1}),$$

$$\lim_{n=\infty}\left\{a_{2,0} S_n^{(0)}(x)+a_{2,1} S_n^{(1)}(x)+a_{2,2} S_n^{(2)}(x)+\cdots\cdots+a_{2,k_2} S_n^{(k_2)}(x)\right\}$$
$$=l(a_{2,0}+a_{2,1}+a_{2,2}+\cdots\cdots+a_{2,k_2}),$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\lim_{n=\infty}\left\{a_{i,0} S_n^{(0)}(x)+a_{i,1} S_n^{(1)}(x)+a_{i,2} S_n^{(2)}(x)+\cdots\cdots+a_{i,k_i} S_n^{(k_i)}(x)\right\}$$
$$=l(a_{i,0}+a_{i,1}+a_{i,2}+\cdots\cdots+a_{i,k_i}),$$

*the necessary and sufficient condition that*

$$\lim_{n=\infty}\left\{b_0 S_n^{(0)}(x)+b_1 S_n^{(1)}(x)+b_2 S_n^{(2)}(x)+\cdots\cdots+b_h S_n^{(h)}(x)\right\}$$
$$=l(b_0+b_1+b_2+\cdots\cdots+b_h)$$

*is that there should be no common root whose real part is not less than* 1, *between equations*

$$a_{1,0}\lambda^{k_1}+a_{1,1}\lambda^{k_1-1}+a_{1,2}\lambda^{k_1-2}+\cdots\cdots+a_{1,k_1}=0,$$
$$a_{2,0}\lambda^{k_2}+a_{2,1}\lambda^{k_2-1}+a_{2,2}\lambda^{k_2-2}+\cdots\cdots+a_{2,k_2}=0,$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$a_{i,0}\lambda^{k_i}+a_{i,1}\lambda^{k_i-1}+a_{i,2}\lambda^{k_i-2}+\cdots\cdots+a_{i,k_i}=0,$$

*which does not make the polynomial*

$$b_0\lambda^h+b_1\lambda^{h-1}+b_2\lambda^{h-2}+\cdots\cdots+b_h$$

*vanish.*

## II.

Prof. G. Pólya ([1]) has recently wanted the proof of a theorem on limit which may be stated as follows:

*If* $$\lim_{n=\infty}(u_1+u_2+u_3+\cdots\cdots+u_n+c\,u_{n+1})=l$$

*where c is a constant, then we have*

$$\lim_{n=\infty}(u_1+u_2+u_3+\cdots\cdots+u_n)=l$$

*when and only when* $$R(c)>\frac{1}{2} \text{ or } c=0.$$

Prof. T. Kubota ([2]) has remarked that this theorem can be easily proved by applying the theorem on limit due to Prof. O. Toeplitz.([3]) I will here show that this theorem can be also proved in the following manner.

Let us put

$$u_1+u_2+\cdots\cdots+u_n+c\,u_{n+1}=l+\varepsilon_n,$$

$$u_1+u_2+u_3+\cdots\cdots+u_n=y_n\,;$$

then we have

$$y_n+c\,(y_{n+1}-y_n)=l+\varepsilon_n,$$

that is, $$(1-c)\,(y_n-l)+c\,(y_{n+1}-l)=\varepsilon_n,$$

we thus obtain

$$(1-c)\,(y_{n+p}-l)+c\,(y_{n+p+1}-l)=\varepsilon_{n+p}.$$

Considering that $c$ is not zero, we add these relations side by side, after having multiplied by $\left(1-\frac{1}{c}\right)^{m-p}$ corresponding to the values $p=0$, $1, 2, \cdots\cdots, m$, then the result becomes

$$c\,(y_{n+m+1}-l)-c\left(1-\frac{1}{c}\right)^{m+1}(y_n-l)=\sum_{p=0}^{p=m}\varepsilon_{n+p}\left(1-\frac{1}{c}\right)^{m-p}.$$

Therefore

$$y_{n+m+1}-l=\left(1-\frac{1}{c}\right)^{m+1}(y_n-l)+\sum_{p=0}^{p=m}\frac{\varepsilon_{n+p}}{c}\left(1-\frac{1}{c}\right)^{m-p}.$$

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([1]) Archiv der Mathematik und Physik (3) 24 (1915).

([2]) This Journal, vol. 9 (1916), pp. 184–185.

([3]) O. Toeplitz, Über allgemeine lineare Mittelbildungen, Prace matematyczno-fizyczne, vol. 22 (1911), pp. 113–119.

Suppose first that the real part of $c$ is greater than $\frac{1}{2}$, namely, the absolute value of $1-\frac{1}{c}$ is less than 1. Then we see

$$\left|y_{n+m+1}-l\right| \leqq \left|\left(1-\frac{1}{c}\right)^{m+1}(y_n-l)\right| + \sum_{p=0}^{p=m}\left|\frac{\varepsilon_{n+p}}{c}\left(1-\frac{1}{c}\right)^{m-p}\right|.$$

Since $\varepsilon_n$ tends to zero as $n$ tends to infinity, we can, when $\varepsilon$ is any small positive quantity, determine a positive integer $n$ such that

$$\left|\frac{\varepsilon_{n+p}}{c}\right| \leqq \frac{\varepsilon}{2}\left\{1-\left|1-\frac{1}{c}\right|\right\}$$

for all the values of $p=0, 1, 2, \cdots\cdots$, and further we can find a positive integer $M$ such that

$$\left|\left(1-\frac{1}{c}\right)^{m+1}(y_n-l)\right| \leqq \frac{\varepsilon}{2} \qquad \text{if} \quad m \geqq M.$$

Consequently we have, for such $n$ and $M$,

$$\left|y_{n+m+1}-l\right| \leqq \frac{\varepsilon}{2}+\frac{\varepsilon}{2}\left(1-\left|1-\frac{1}{c}\right|\right)\sum_{p=0}^{p=m}\left|1-\frac{1}{c}\right|^{m-p}$$

$$< \frac{\varepsilon}{2}+\frac{\varepsilon}{2}\left(1-\left|1-\frac{1}{c}\right|\right)\sum_{p=0}^{p=\infty}\left|1-\frac{1}{c}\right|^{p},$$

i. e. $\qquad |y_{n+m+1}-l| < \varepsilon \qquad$ if $\quad m \geqq M.$

Thus, by the definition of limit, we see that

$$\lim_{n=\infty} y_n = l.$$

Therefore $\qquad \lim_{n=\infty}(u_1+u_2+u_3+\cdots\cdots+u_n)=l.$

Secondly we consider that the real part of $c$ is not greater than $\frac{1}{2}$, viz. the absolute value of $1-\frac{1}{c}$ is not less than 1. For the purpose of verifying that this theorem is generally untrue, we take the following example.

Let us put

$$u_1 = l-c,$$

$$u_{p+1}=\left(1-\frac{1}{c}\right)^{p-1}, \qquad p=1, 2, 3, \cdots\cdots$$

then we have

$$u_1+u_2+u_3+\cdots\cdots+u_n=l-c+\sum_{p=0}^{p=n-2}\left(1-\frac{1}{c}\right)^p$$
$$=l-c+c\left\{1-\left(1-\frac{1}{c}\right)^{n-1}\right\},$$

and so
$$u_1+u_2+u_3+\cdots\cdots+u_n=l-c\left(1-\frac{1}{c}\right)^{n-1},$$

It follows that

$$u_1+u_2+u_3+\cdots\cdots+u_n+c\,u_{n+1}=l,$$

whence it is evident that

$$\lim_{n=\infty}(u_1+u_2+u_3+\cdots\cdots+u_n+c\,u_{n+1})=l.$$

While since the absolute value of $1-\frac{1}{c}$ is not less than 1, $\left(1-\frac{1}{c}\right)^{n-1}$ does not tend to zero as $n$ increases indefinitely, the series $\Sigma\, u_n$ does not converge to $l$.

The proof of the theorem proposed by Pólya is thus completed.

## III.

Applying Toeplitz's theorem Prof. T. Kubota (1) has proved a generalization of Pólya's theorem, which may be stated as follows:

The necessary and sufficient condition that, whenever

$$\lim(u_1+u_2+\cdots\cdots+u_n+c_1\,u_{n+1}+c_2\,u_{n+2}+\cdots\cdots+c_p\,u_{n+p})=l,$$

the series $\Sigma\, u_n$ should converge to $l$ is that all the roots of the equation

$$c_p\,\lambda^p+(c_{p-1}-c_p)\,\lambda^{p-1}+\cdots\cdots+(1-c_1)=0$$

should lie within the unit circle.

In this Journal the second proof (2) of this theorem has been completed by using a method similar to that employed by me.

We see by a slight modification that this theorem is equivalent to the following:

When

$$\lim_{n=\infty}\left\{a_0\,y_n+a_1\,y_{n+1}+a_2\,y_{n+2}+\cdots\cdots+a_k\,y_{n+k}\right\}$$
$$=l\,(a_0+a_1+a_2+\cdots\cdots+a_k),$$

(1) T. Kubota, Some applications of a theorem of Toeplitz's on limits, Tôkyô Butsurigakko Zasshi. No. 295, (1916), pp. 243–246.

(2) T. Kubota, Ein Satz über den Grenzwert, this Journal, vol. 12 (1917), pp. 222–224.

the necessary and sufficient condition that the sequence $y_n$ should converge to the limit $l$ is that the equation

$$a_0+a_1\lambda+a_2\lambda^2+\cdots\cdots+a_k\lambda^k=0$$

should have all its roots within the unit circle.

If we start with this theorem and proceed in a manner just similar to that in I, we arrive at the following theorem:

Let the sequence $y_1, y_2, y_3, \cdots\cdots, y_n, \cdots\cdots$ have such a nature that

$$\lim_{n=\infty}\left\{a_{1,0}\,y_n+a_{1,1}\,y_{n+1}+a_{1,2}\,y_{n+2}+\cdots\cdots+a_{1,k_1}\,y_{n+k_1}\right\}$$
$$=l\,(a_{1,0}+a_{1,1}+a_{1,2}+\cdots\cdots+a_{1,k_1}),$$

$$\lim_{n=\infty}\left\{a_{2,0}\,y_n+a_{2,1}\,y_{n+1}+a_{2,2}\,y_{n+2}+\cdots\cdots+a_{2,k_2}\,y_{n+k_2}\right\}$$
$$=l\,(a_{2,0}+a_{2,1}+a_{2,2}+\cdots\cdots+a_{2,k_2}),$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\lim_{n=\infty}\left\{a_{i,0}\,y_n+a_{i,1}\,y_{n+1}+a_{i,2}\,y_{n+2}+\cdots\cdots+a_{i,k_i}\,y_{n+k_i}\right\}$$
$$=l\,(a_{i,0}+a_{i,1}+a_{i,2}+\cdots\cdots+a_{i,k_i}),$$

then the necessary and sufficient condition that the relation

$$\lim_{n=\infty}\left\{b_0\,y_n+b_1\,y_{n+1}+b_2\,y_{n+2}+\cdots\cdots+b_h\,y_{n+h}\right\}$$
$$=l\,(b_0+b_1+b_2+\cdots\cdots+b_h)$$

may be ensured is that there is no common root between the equations

$$f_1(\lambda)=a_{1,0}+a_{1,1}\lambda+a_{1,2}\lambda^2+\cdots\cdots+a_{1,k_1}\lambda^{k_1}=0,$$
$$f_2(\lambda)=a_{2,0}+a_{2,1}\lambda+a_{2,2}\lambda^2+\cdots\cdots+a_{2,k_2}\lambda^{k_2}=0,$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$f_i(\lambda)=a_{i,0}+a_{i,1}\lambda+a_{i,2}\lambda^2+\cdots\cdots+a_{i,k_i}\lambda^{k_i}=0,$$

which does not make the polynomial

$$g(\lambda)=b_0+b_1\lambda+b_2\lambda^2+\cdots\cdots+b_h\lambda^h$$

vanish, and lies outside or on the unit circle.

In the last theorem if we put

$$y_n=u_1+u_2+u_3+\cdots\cdots+u_n$$

$$a_{1,0}=1-c_{1,1},\ a_{1,1}=c_{1,1}-c_{1,2},\cdots\cdots a_{1,k_1-1}=c_{1,k_1-1}-c_{1,k_1},\ a_{1,k_1}=c_{1,k_1},$$
$$a_{2,0}=1-c_{2,1},\ a_{2,1}=c_{2,1}-c_{2,2},\cdots\cdots a_{2,k_2-1}=c_{2,k_2-1}-c_{2,k_2},\ a_{2,k_2}=c_{2,k_2},$$
$$\cdots\cdots\cdots\cdots\cdots\cdots$$
$$a_{i,0}=1-c_{i,1},\ a_{i,1}=c_{i,1}-c_{i,2},\cdots\cdots a_{i,k_i-1}=c_{i,k_i-1}-c_{i,k_i},\ a_{i,k_i}=c_{i,k_i},$$
$$b_0=1-d_1,\qquad b_1=d_1-d_2,\cdots\cdots\qquad b_{h-1}=d_{h-1}-d_h,\ b_h=d_h,$$

then we can change the statement of this theorem as follows:

When all the sequences

$$u_1+u_2+\cdots\cdots+u_n+c_{1,1}\,u_{n+1}+c_{1,2}\,u_{n+2}+\cdots\cdots+c_{1,k_1}\,u_{n+k_1},$$
$$u_1+u_2+\cdots\cdots+u_n+c_{2,1}\,u_{n+1}+c_{2,2}\,u_{n+2}+\cdots\cdots+c_{2,k_2}\,u_{n+k_2},$$
$$\cdots\cdots\cdots\cdots\cdots\cdots$$
$$u_1+u_2+\cdots\cdots+u_n+c_{i,1}\,u_{n+1}+c_{i,2}\,u_{n+2}+\cdots\cdots+c_{i,k_i}\,u_{n+k_i}$$

converge to the same limit $l$ as $n$ increases indefinitely, then the necessary and sufficient condition that the sequence

$$u_1+u_2+\cdots\cdots+u_n+d_1\,u_{n+1}+d_2\,u_{n+2}+\cdots\cdots+d_h\,u_{n+h}$$

should also converge to the limit $l$ is that there is no common root of the equations

$$c_{1,k_1}\,\lambda^{k_1}+(c_{1,k_1-1}-c_{1,k_1})\,\lambda^{k_1-1}+\cdots\cdots+(1-c_{1,1})=0,$$
$$c_{2,k_2}\,\lambda^{k_2}+(c_{2,k_2-1}-c_{2,k_2})\,\lambda^{k_2-1}+\cdots\cdots+(1-c_{2,1})=0,$$
$$\cdots\cdots\cdots\cdots\cdots\cdots$$
$$c_{i,k_i}\,\lambda^{k_2}+(c_{i,k_i-1}-c_{i,k_i})\,\lambda^{k_i-1}+\cdots\cdots+(1-c_{i,1})=0$$

which does not make the polynomial

$$d_h\,\lambda^h+(d_{h-1}-d_h)\,\lambda^{h-1}+\cdots\cdots+(1-d_1)$$

vanish and lies outside or on the unit circle.

Since this theorem may be proved in a way quite similar to that in I, I give its statement only.

# Einige Sätze den Grenzwert betreffend,

von

TADAHIKO KUBOTA in Sendai.

Der L'Hospital-Stolzsche (1) Satz über die unbestimmten Formen lautet: Wenn $\psi(x)$ mit $x$ beständig gegen $+\infty$ wächst, dann

$$\lim_{n\to\infty}\frac{\varphi(x)}{\psi(x)}=\lim_{n\to\infty}\frac{\varphi'(x)}{\psi'(x)},$$

unter der Voraussetzung, dass der zweite Grenzwert existiert. Es ist aber eine Frage, unter welcher Bedingung man aus der Existenz des ersten Grenzwertes die obige Beziehung ableiten kann. Im Folgenden möchte ich einige hinreichende Bedingungen dafür geben.

Satz 1. *Es sei $g(x)$ eine stetige beständig nichtnegative Funktion und sei*

$$x\div\int_0^x g(x)\,dx$$

*beschränkt; wenn nun*

$$\operatorname*{Lim}_{n\to\infty}\left\{\int_0^x g(x)\left[\int_0^x f(x)\,dx\right]dx\div\int_0^x g(x)\,dx\right\}$$

*existiert und einem endlichen Werte $a$ gleich ist und*

$$\int_0^\infty f(x)^2\left[\int_0^x g(x)\,dx\right]dx$$

*konvergiert, dann*

$$\operatorname*{Lim}_{x\to\infty}\int_0^x f(x)\,dx=a.$$

Zum Beweise setze man

$$F(x)=\int_0^x g(x)\left[\int_0^x f(x)\,dx\right]dx,$$

(1) L'Hospital, Analyse des Infiniments Petits, Paris, 1696;
Stolz, Math. Ann. 14, 15, 23;
Osgood, Ann. of Math. 12, 1897;
Young, Proc. London Math. Soc. 8.

$$\Phi(x)=\int_0^x f(\xi)\left[\int_\xi^x g(x)\,dx\right]d\xi.$$

Differenziert man die beiden Funktionen nach $x$, so erhält man

$$F'(x)=g(x)\int_0^x f(x)\,dx,$$

$$\Phi'(x)=g(x)\int_0^x f(\xi)\,d\xi,$$

woraus sich ergibt

$$F'(x)=\Phi'(x).$$

Da aber $$F(0)=\Phi(0)=0,$$

so bekommt man

$$F(x)=\Phi(x).$$

Nun

$$\int_0^x g(x)\left[\int_0^x f(x)\,dx\right]dx\div\int_0^x g(x)\,dx-\int_0^x f(x)\,dx$$

$$=\left\{\int_0^x f(\xi)\left[\int_\xi^x g(x)\,dx\right]d\xi-\int_0^x f(\xi)\left[\int_0^x g(x)\,dx\right]d\xi\right\}\div\int_0^x g(x)\,dx$$

$$=-\int_0^x f(\xi)\left[\int_0^\xi g(\xi)\,d\xi\right]d\xi\div\int_0^x g(x)\,dx.$$

Der Beweis besteht also darin zu zeigen, dass

$$\operatorname*{Lim}_{x\to\infty}\left\{\int_0^x f(\xi)\left[\int_0^\xi g(\xi)\,d\xi\right]d\xi\div\int_0^x g(x)\,dx\right\}=0.$$

Da aber

$$\int_0^\infty f(\xi)^2\left[\int_0^\xi g(\xi)\,d\xi\right]d\xi$$

konvergiert, so kann man eine positive Zahl $l$ so gross wählen, dass

$$\int_l^\infty f(\xi)^2\left[\int_0^\xi g(\xi)\,d\xi\right]d\xi<\varepsilon^2,$$

wobei $\varepsilon$ eine beliebig gewählte kleine positive Zahl bedeutet. Nach der Schwarzschen Ungleichung für $x>l$

$$\left|\int_l^x f(\xi)\left[\int_0^\xi g(\xi)\,d\xi\right]d\xi\right|$$

$$\leqq \sqrt{\int_l^x d\xi \int_0^\xi g(\xi)\,d\xi \,.\, \int_l^x f(\xi)^2 \left[\int_0^\xi g(\xi)\,d\xi\right] d\xi}$$

$$\leqq \varepsilon \sqrt{\int_l^x d\xi \int_0^x g(\xi)\,d\xi} \leqq \varepsilon \sqrt{\int_0^x d\xi \int_0^x g(\xi)\,d\xi}$$

$$\leqq \varepsilon \int_0^x g(x)\,dx \sqrt{\frac{x}{\int_0^x g(x)\,dx}}.$$

Bezeichnet man die obere Grenze von

$$x \div \int_0^x g(x)\,dx$$

mit $L$, so ist

$$\left| \int_l^x f(\xi) \left[\int_0^\xi g(\xi)\,d\xi\right] d\xi \div \int_0^x g(x)\,dx \right| \leqq \varepsilon L,$$

und

$$\left| \int_0^x f(\xi) \left[\int_0^\xi g(\xi)\,d\xi\right] d\xi \div \int_0^x g(x)\,dx \right|$$

$$\leqq \left| \int_0^l f(\xi) \left[\int_0^\xi g(\xi)\,d\xi\right] d\xi \div \int_0^x g(x)\,dx \right|$$

$$+ \left| \int_l^x f(\xi) \left[\int_0^\xi g(\xi)\,d\xi\right] d\xi \div \int_0^x g(x)\,dx \right|.$$

Da aber

$$\lim_{x \to \infty} \int_0^x g(x)\,dx = \infty,$$

so kann man $x$ so gross wählen, dass

$$\left| \int_0^l f(\xi) \left[\int_0^\xi g(\xi)\,d\xi\right] d\xi \div \int_0^x g(x)\,dx \right| < \varepsilon$$

und folglich

$$\left| \int_0^x f(\xi) \left[\int_0^\xi g(\xi)\,d\xi\right] d\xi \div \int_0^x g(x)\,dx \right| < \varepsilon (1 + \sqrt{L}).$$

Schliesslich

$$\operatorname*{Lim}_{x \to \infty} \left\{ \int_0^x f(\xi) \left[\int_0^\xi g(\xi)\,d\xi\right] d\xi \div \int_0^x g(x)\,dx \right\} = 0$$

und daraus ergibt sich

$$\lim_{x\to\infty}\int_0^x f(x)\,dx = a.$$

Damit ist der Satz bewiesen.

Als Spezialfall des obigen Satzes für $g(x)=1$ erhält man den

Satz 2. *Wenn*

$$\lim_{x\to\infty}\left[\int_0^x dx\int_0^x f(x)\,dx \div x\right]$$

*existiert und einem endlichen Wert $a$ gleich ist und*

$$\int_0^\infty f(x)^2\,x\,dx$$

*konvergiert, dann*

$$\lim_{x\to\infty}\int_0^x f(x)\,dx = a.$$

Dieser Satz ist das Analogon des Fejérschen Satzes ([1]) für die unendlichen Reihen. Den Grundgedanke dieser Beweisführung verdanke ich den Herren Dr. Fejér und Dr. Pólya.

Satz 3. *Wenn $\varphi(x)$ zweimal stetig differenzierbar ist und*

$$\lim_{x\to\infty}\frac{\varphi(x)}{x}$$

*existiert und einem endlichen Wert $a$ gleich ist und*

$$\int_0^\infty \varphi''(x)^2\,x\,dx$$

*konvergiert, dann*

$$\lim_{x\to\infty}\varphi'(x) = a.$$

Es ist möglich eine zweimal stetig differenzierbare Funktion $F(x)$ zu finden welche den folgenden Bedingungen genügt:

(i) $\qquad F(x)=\varphi(x)$ für $x \geqq k > 0,$

(ii) $\qquad F(0)=0, \quad F'(0)=0.$

Da nun nach der Voraussetzung

$$\int_0^\infty \varphi''(x)^2\,x\,dx$$

([1]) Schwarz, Festschriften, 1914.

konvergiert, so konvergiert auch

$$\int_0^\infty F''(x)^2\, x\, dx,$$

und

$$\operatorname*{Lim}_{x\to\infty} \frac{1}{x}\int_0^x dx \int_0^x F''(x)\, dx = \operatorname{Lim} \frac{1}{x}\int_0^x F'(x)\, dx$$

$$= \operatorname*{Lim}_{x\to\infty} \frac{1}{x} F(x) = \operatorname*{Lim}_{x\to\infty} \frac{1}{x}\varphi(x) = a.$$

Nach dem Satze 2 gilt

$$\operatorname*{Lim}_{x\to\infty} \int_0^x F''(x)\, dx = a,$$

d.h.

$$\operatorname*{Lim}_{x\to\infty} F'(x) = a.$$

Damit ist bewiesen, dass

$$\operatorname*{Lim}_{x\to\infty} \varphi'(x) = a.$$

Satz 4. *Es seien $F(x)$, $G(x)$ zweimal stetig differenzierbare Funktionen, von denen $G(x)$ beständig positiv und gegen $+\infty$ beständig wächst; wenn nun*

$$\operatorname*{Lim}_{x\to\infty} \frac{F(x)}{G(x)}$$

*existiert und einem endlichen Werte $a$ gleich ist und*

$$\int_0^\infty \frac{G(x)}{G'(x)}\left(\frac{d}{dx}\,\frac{F'(x)}{G'(x)}\right)^2 dx$$

*konvergiert, dann*

$$\operatorname*{Lim}_{x\to\infty} \frac{F'(x)}{G'(x)} = a.$$

Zum Beweise setze man

$$G(x) = y.$$

Die Umkehrung dieser Gleichung sei

$$x = G^{-1}(y) = x(y).$$

Dann

$$\operatorname*{Lim}_{x\to\infty} \frac{F(x(y))}{y} = \operatorname*{Lim}_{x\to\infty} \frac{F(x)}{G(x)} = a \tag{1}$$

Aus der Konvergenz des Integrals

$$I=\int_0^\infty \frac{G(x)}{G'(x)}\left(\frac{d}{dx}\frac{F'(x)}{G'(x)}\right)^2 dx,$$

durch Transformation

$$x=x(y),\quad y=G(x),\quad dy=G'(x)\,dx,$$

folgt die Konvergenz des Integrals

$$I=\int_0^\infty y\left\{G'(x(y))\frac{d}{dy}\frac{F'(x(y))}{G'(x(y))}\right\}^2\frac{dy}{G'(x(y))^2}$$

$$=\int_0^\infty y\left\{\frac{d}{dy}\frac{F'(x(y))}{G'(x(y))}\right\}^2 dy$$

$$=\int_0^\infty y\left\{\frac{d^2}{dy^2}F(x(y))\right\}^2 dy.$$

Daraus und aus (1) folgt nach dem Satze (3), dass

$$\lim_{y\to\infty}\frac{d}{dy}F(x(y))=a,$$

$$\lim_{y\to\infty}F'(x(y))\,x'(y)=a,$$

d.h.

$$\lim_{y\to\infty}\frac{F'(x(y))}{G'(x(y))}=a.$$

Schliesslich gelangen wir zum Resultate

$$\lim_{x\to\infty}\frac{F'(x)}{G'(x)}=a.$$

Satz 5. *Wenn*

$$\lim_{x\to\infty}\frac{1}{x}\int_0^x dx\int_0^x f(x)\,dx$$

*existiert und einem endlichen Werte $a$ gleich ist und für hinreichend grosses $x$*

$$xf(x)>-A$$

*wobei $A$ eine von $x$ unabhängige positive Zahl bedeutet, dann*

$$\lim_{x\to\infty}\int_0^x f(x)\,dx=a.$$

Dies ist zu dem Hardy-Landauschen ([1]) Satze für die unendlichen Reihen ganz analog und lässt sich in ähnlicher Weise wie folgt beweisen.

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([1]) G. H. Hardy, Proc. London Math. Soc. 8.
E. Landau, Prace mat. fiz. 21.; de la Vallée Poussin, Cours d'analyse infini. II.
J. E. Littlewood, Proc. London Math. Soc. 9.
E. Landau, Rendiconti del Circolo Mat. di Palermo, 1913.
E. Landau, Münchener Berichte, 1913.

Man setze

$$S(x)=\int_0^x f(x)\,dx,\quad \int_0^x dx\int_0^x f(x)\,dx=x\,\sigma(x).$$

Für eine positive Zahl $l$, welche kleiner als $x$ ist,

$$\int_{x-l}^{x} S(x)\,dx=x\,\sigma(x)-(x-l)\,\sigma(x-l)=l\,\sigma(x-l)+x\,[\sigma(x)-\sigma(x-l)],$$

$$\int_{x}^{x+l} S(x)\,dx=(x+l)\,\sigma(x+l)-x\,\sigma(x)=l\,\sigma(x+l)+x\,[\sigma(x+l)-\sigma(x)].$$

Wenn $\xi$ zwischen $x$ und $x-l$ liegt,

$$S(\xi)=S(x)-\int_{\xi}^{x} f(x)\,dx<S(x)+\frac{A}{x-l}l.$$

Wenn $\eta$ zwischen $x$ und $x+l$ liegt,

$$S(\eta)=S(x)+\int_{x}^{\eta} f(x)\,dx>S(x)-l\frac{A}{x}.$$

Setzt man die obere und untere Schranke in die ersten beiden Beziehungen ein, so bekommt man

$$S(x)+\frac{Al}{x-l}>\sigma(x-l)+\frac{x}{l}[\sigma(x)-\sigma(x-l)],$$

$$S(x)-\frac{l}{x}A<\sigma(x+l)+\frac{x}{l}[\sigma(x+l)-\sigma(x)].$$

Nun ist $a$ der Grenzwert von $\sigma(x)$. Wenn also $x$ und $x-l$ gegen $+\infty$ wachsen, sodass $\frac{l}{x}$ sich dem Werte $\varepsilon$ als Grenze nähert, wobei $\varepsilon$ eine beliebige positive Zahl bedeutet, dann

$$\underline{\mathrm{Lim}}\, S(x)+\frac{\varepsilon A}{1-\varepsilon}\geqq a,$$

$$\overline{\mathrm{Lim}}\, S(x)-\varepsilon A\leqq a.$$

Damit ist bewiesen, dass

$$\underline{\mathrm{Lim}}\, S(x)=\overline{\mathrm{Lim}}\, S(x)=a,\quad \text{w. z. b. w.}$$

Satz 6. *Wenn $\varphi(x)$ zweimal stetig differenzierbar ist und*

$$\lim_{x\to\infty}\frac{\varphi(x)}{x}$$

*existiert und einem endlichen Werte a gleich ist und für hinreischend grosses x*

$$x\,\varphi''(x) > -A,$$

*wobei A eine von x unabhängige endliche positive Zahl bedeutet, dann*

$$\lim_{x\to\infty} \varphi'(x) = a.$$

Es sei $F(x)$ eine zweimalstetig differenzierbare Funktion, die den Bedingungen (i), (ii) genügt:

(i) $\qquad F(x) = \varphi(x) \qquad$ für $\quad x \geqq k > 0,$

(ii) $\qquad F(0) = F'(0) = 0.$

Nun wenden wir den Satz 7 auf die Funktion

$$F(x) = \int_0^x dx \int_0^x F''(x)\,dx$$

an, dann

$$\lim_{x\to\infty} \frac{1}{x}\int_0^x dx \int_0^x F''(x)\,dx = \lim_{x\to\infty} \frac{F(x)}{x} = \lim_{x\to\infty} \frac{\varphi(x)}{x} = a,$$

während $\qquad x\,F''(x) > -A',$

wobei $A'$ eine von $x$ unabhängige endliche positive Zahl bedeutet. Daraus folgt

$$\lim_{x\to\infty} F'(x) = \lim_{x\to\infty} \int_0^x F''(x)\,dx = a,$$

d.h. $\qquad \lim_{x\to\infty} \varphi'(x) = a.$

Satz 7. *Es seien $F(x)$, $G(x)$ zweimal stetig differenzierbare Funktionen von denen $G(x)$ beständig gegen $+\infty$ wächst und wenn*

$$\lim_{x\to\infty} \frac{F(x)}{G(x)}$$

*existiert und einem endlichen Werte a gleich ist und*

$$\frac{G(x)}{G'(x)} \frac{d}{dx} \frac{F'(x)}{G'(x)} > -A,$$

*für hinreichend grosses x gilt, wobei A eine von x unabhängige endliche positive Zahl bedeutet, dann*

$$\operatorname*{Lim}_{x\to\infty} \frac{F'(x)}{G'(x)} = \alpha.$$

Zum Beweise setze man

$$y = G(x), \qquad x = G^{-1}(y) = x(y),$$

$$\operatorname*{Lim}_{y\to\infty} \frac{F(x(y))}{y} = \operatorname*{Lim}_{y\to\infty} \frac{F(x(y))}{G(x(y))} = \operatorname*{Lim}_{x\to\infty} \frac{F(x)}{G(x)} = \alpha.$$

$$y\frac{d^2}{dy^2}F(x(y)) = y\frac{d}{dy}[F'(x(y))\,x'(y)] = G(x)\frac{d}{G'(x)\,dx}\left[\frac{F'(x)}{G'(x)}\right].$$

Da aber

$$\frac{G(x)}{G'(x)}\,\frac{d}{dx}\,\frac{F'(x)}{G'(x)} > -A$$

gilt, so bekommt man

$$y\frac{d^2}{dy^2}F(x(y)) > -A.$$

Folglich bekommt man

$$\operatorname*{Lim}_{y\to\infty} \frac{d}{dy}F(x(y)) = \alpha,$$

d.h.

$$\operatorname*{Lim}_{y\to\infty} F'(x(y))\,x'(y) = \alpha.$$

Schliesslich gelangen wir zum Resultat

$$\operatorname*{Lim}_{x\to\infty} \frac{F'(x)}{G'(x)} = \operatorname*{Lim}_{x\to\infty} \frac{F'(x(y))}{G'(x(y))} = \operatorname*{Lim}_{y\to\infty} F'(x(y))\,x'(y) = \alpha.$$

Damit ist der Satz bewiesen.

# Über summierbare Reihen und Integrale,

von

MATSUSABURÔ FUJIWARA in Sendai.

In der Arbeit: Über einige Sätze den Grenzwert betreffend, dieses Journal, 15, 1919, S. 314, hat Herr Prof. Kubota den Hardy-Landauschen Satz über summierbare Reihen auf den Fall der summierbaren Integrale ausgedehnt. Diese Arbeit hat meine Aufmerksamkeit auf den folgenden Landauschen Hilfssatz ([1]) gelenkt:

Es sei $g(x)$ für $0<x<1$ reell und zweimal differentierbar; ebenda sei

$$(1-x)^2\,|\,g''(x)\,|<K.$$

Für $x\to 1$ sei $g(x)\to A$. Dann ist für $x\to 1$, $(1-x)g'(x)\to 0$.

Ich habe dabei bemerkt, dass man den Kubotaschen Satz durch die Landausche Beweisführung aus der Taylorschen Formel ableiten kann. Verfolgend den Grundgedanken, welcher in dem obigen Hilfssatze liegt, konnte ich weiter den Kubotaschen und den Hardy-Landauschen Sätze auf Doppelintegrale bzw. Doppelreihe verallgemeinern. Dies werde ich in folgenden Zeilen darlegen.

**1.** Es sei $f(x)$ für $x\geqq 0$ reell und zweimal differentierbar, und setze voraus

$$f(x)/x\to A,\quad \text{für } x\to\infty\,;$$

$$xf''(x)>-M,\ \text{für } x\geqq 0,$$

wo $M$ eine positive Konstante bedeutet. Aus der Taylorschen Formel erhält man

$$\frac{f(\alpha x)-f(x)}{\alpha x-x}-f'(x)=\frac{\alpha-1}{2\beta}\,\beta x f''(\beta x),$$

$$\alpha>0,\quad \beta=1+\theta(\alpha-1),\quad 0<\theta<1.$$

Weil der erste Glied in der linken Seite für $x\to\infty$ den Grenzwert $A$ besitzt, und $\beta>1$ oder $<1$ je nachdem $\alpha>\!<1$ ist, kann man aus der Annahme $xf''(x)>-M$ schliessen:

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([1]) Landau: Über einen Satz des Herrn Littlewood, R. Palermo, 35, 1913.

$$A-\lim_{x\to\infty} \sup. f'(x) > -(\alpha-1)M/2, \text{ für } \alpha>1,$$

$$A-\lim_{x\to\infty} \inf. f'(x) < -(\alpha-1)M/2, \text{ für } \alpha<1;$$

daraus folgt, da $|\alpha-1|$ beliebig klein gewählt werden kann,

$$A \geqq \lim \sup f'(x), \quad A \leqq \lim \inf f'(x),$$

d.h.

$$f'(x) \to A \text{ für } x\to\infty.$$

Also erhält man den Kubotaschen Satz:

*Es sei $f(x)$ für $x \geqq 0$ reell und zweimal differentierbar. Ist $f(x)/x \to A$ für $x\to\infty$, und $xf''(x) > -M$ für $x \geqq 0$, so ist $f'(x) \to A$ für $x\to\infty$.*

Setzt man darin

$$f(x) = \int_0^x dx \int_0^x \varphi(x)dx,$$

wo $\varphi(x)$ für jede endliche Intervalle in $(0, \infty)$ integrabel ist, so erhält man den zu dem Hardy-Landauschen Satz entsprechenden Satz des Herrn Kubota:

*Ist*

$$\int_0^\infty \varphi(x)dx$$

*summierbar* $(C1)$ *und gleich $A$, und $x\varphi(x) > -M$, $(M>0)$ für $x \geqq 0$, so ist dieses Integral konvergent und zwar gleich $A$.*

**2.** Aus ähnlicher Überlegung kann man die folgenden Sätze ableiten.

*Es sei $f(x)$ für $0<x<a$ reell und zweimal differentierbar, und ebenda sei $xf''(x) > -M$. Ist $f(x)/x \to A$ für $x\to 0$, so ist $f'(x) \to A$ für $x\to 0$.*

*Es sei $f(x)$ für $0<x<1$ reell und zweimal differentierbar, und ebenda $x^2f''(x) > -M$. Ist $f(x) \to A$ für $x\to 0$, so ist $xf'(x) \to 0$ für $x\to 0$.*

*Es sei $f(x)$ für $0<x<1$ reell und zweimal differentierbar, und ebenda $(1-x)^2f''(x) > -M$. Ist $f(x) \to A$ für $x\to 1$, so ist $(1-x)f'(x) \to 0$ für $x\to 1$* ([1]).

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([1]) Dieser Satz wurde von den Herren Hardy und Littlewood auf andere Weise bewiesen, in der Arbeit: Tauberian Theorems on power series with positive coefficients etc., Proc. London Math. Soc., (2) 13, 1914, p. 188.

Der letztere Satz ist eine Verallgemeinerung des Landauschen Hilfssatzes.

Natürlich kann man in dieser Reihe der Sätze die Bedingung $(>-M)$ durch $(<M)$ ersetzen.

Ferner erhält man aus der Taylorschen Formel den folgenden verallgemeinerten Littlewoodschen Hilfssatz:

*Es sei $f(x)$ für $0<x<1$ reell und beliebig oft differentierbar, und sei ebenda $x^r f^{(r)}(x)$ (oder $(1-x)^r f^{(r)}(x)$) für jedes feste positive ganze $r$ beschränkt oben oder unten. Ist für $x\to 0$ (oder $x\to 1$) $f(x)\to A$, so ist für jedes positive ganze $r$*

$$x^r f^{(r)}(x)\to 0 \quad (\text{oder } (1-x)^r f^{(r)}(x)\to 0.)$$

**3.** Auf demselben Standpunkt stehend werden wir zunächst dem Hardy-Landauschen ([1]) Satze einen neuen Beweis geben, wenn auch dies im Grunde nicht von dem Landauschen verschieden, und der Letztere in Einfachheit nicht zu wünschen übrig ist.

Es seien

$$f(n)=s_0+s_1+\cdots+s_n, \quad s_n=a_0+a_1+\cdots+a_n,$$
$$\Delta f(n)=f(n)-f(n-1)=s_n,$$
$$\Delta^2 f(n)=\Delta f(n)-\Delta f(n-1)=a_n.$$

Dann gilt die Relation

$$\begin{aligned} f(m)-f(n)-(m-n)\,\Delta f(n) \\ &= \Delta^2 f(m)+2\,\Delta^2 f(m-1)+3\,\Delta^2 f(m-2)+\cdots\cdots \\ &\quad +(m-n)\,\Delta^2 f(n+1), \quad \text{für } m>n, \\ &= \Delta^2 f(m+2)+2\,\Delta^2 f(m+3)+3\,\Delta^2 f(m+4)+\cdots\cdots \\ &\quad +(n-m-1)\,\Delta^2 f(n), \quad \text{für } n>m. \end{aligned}$$

Diese Relation spielt hier dieselbe Rolle wie die Taylorsche Formel im vorigen Falle.

Setzt man voraus, dass

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([1]) Hardy: The ordinal relations of the terms of a convergent sequence, Proc. London Math. Soc. (2) 8, 1910.

Landau: Über die Bedeutung einiger neuen Grenzwertsatze der Herren Hardy und Axer, Prace matematyczno-fizyczne, 21, 1910.

Diese letztere Arbeit ist mir leider unzugänglich; deswegen habe ich den Landauschen Beweise in Vallée Poussinschen Cours d'Analyse infinitésimale, t. 2, kennen gelernt.

$$f(n)/n \to A \quad \text{für} \quad n \to \infty,$$

$$n\Delta^2 f(n) = na_n > -M, \quad (M>0),$$

so wird für $m>n$

$$\frac{f(m)-f(n)}{m-n} - \Delta f(n) > -\frac{M}{m-n}\left(\frac{1}{m} + \frac{2}{m-1} + \cdots\cdots + \frac{m-n}{n+1}\right),$$

$$> -\frac{M(m-n+1)}{2n},$$

und für $n>m$

$$\frac{f(m)-f(n)}{m-n} - \Delta f(n) < \frac{M}{n-m}\left(\frac{1}{m+2} + \frac{2}{m+3} + \cdots\cdots + \frac{n-m-1}{n}\right)$$

$$< \frac{M(n-m)}{2m}.$$

Lässt man $m$, $n \to \infty$ derart dass $\frac{m}{n} \to \varepsilon$, so wird

$$A - \limsup_{n\to\infty} \Delta f(n) \geqq -M(\varepsilon-1)/2, \quad \text{für} \quad \varepsilon>1,$$

$$A - \liminf_{n\to\infty} \Delta f(n) \leqq M(1-\varepsilon)/2\varepsilon, \quad \text{für} \quad \varepsilon<1.$$

Da $\varepsilon$ ganz beliebig und $|\varepsilon-1|$ beliebig klein gewählt werden kann, so hat man

$$A \geqq \limsup \Delta f(n),$$

$$A \leqq \liminf \Delta f(n),$$

d.h.

$$\Delta f(n) = s_n \to A \quad \text{für} \quad n \to \infty.$$

Damit ist der Hardy-Landausche Satz bewiesen:

*Ist* $\sum_{n=0}^{\infty} a_n$ *summierbar* $(C1)$ *und die Summe gleich* $A$, *so ist* $\Sigma a_n$ *unter der Annahme* $na_n > -M$ *konvergent und gleich* $A$.

Der folgende Satz ist auch in gleicher Weise nachweisbar.

Es sei $f(x)$ für $x \geqq 0$ reell und stetig. Ist $f(x)/x \to A$ für $x \to \infty$, und $x\Delta^2 f(x) > -M$, so ist $\Delta f(x) \to A$ für $x \to \infty$.

**4.** Wir betrachten nunmehr den Fall der Doppelreihen.

Es seien

$$s_{m,n}=\sum_{\mu=1}^{m}\sum_{\nu=1}^{n}a_{\mu,\nu}$$

$$S_{m,n}=\sum_{\mu=1}^{m}\sum_{\nu=1}^{n}s_{\mu,\nu},$$

$$s_{m,n}-s_{m-1,n}=p_{m,n}=\sum_{\nu=1}^{n}a_{m,\nu},\quad s_{m,n}-s_{m,n-1}=q_{m,n}=\sum_{\mu=1}^{m}a_{\mu,n},$$

und setze voraus

$$\frac{S_{m,n}}{mn}\to A \quad \text{für} \quad m,n\to\infty$$

und

$$mp_{m,n}>-M,\quad nq_{m,n}>-M,\quad (M>0).$$

Ist $m>\mu$, $n>\nu$, dann ist

$$S_{m,n}-S_{m,\nu}-S_{\mu,\nu}+S_{\mu,\nu}=\sum_{i=\mu+1}^{m}\sum_{k=\nu+1}^{n}s_{i,k},$$

$$s_{i,k}=s_{\mu,\nu}+\sum_{r=\mu+1}^{i}p_{r,k}+\sum_{t=\nu+1}^{k}q_{\mu,t},$$

und folglich

$$\begin{aligned}(S_{m,n}-S_{m,\nu}-S_{\mu,n}+S_{\mu,\nu})-(m-\mu)(n-\nu)\,S_{\mu,\nu}\\ =\sum_{i=\mu+1}^{m}\sum_{k=\nu+1}^{n}\Big(\sum_{r=\mu+1}^{i}p_{r,k}+\sum_{t=\nu+1}^{k}q_{\mu,t}\Big)\\ >-\sum_{i=\mu+1}^{m}\sum_{k=\nu+1}^{n}M\left(\frac{m-\mu}{\mu}+\frac{n-\nu}{\nu}\right).\end{aligned}$$

Deswegen, wenn man $\mu$, $\nu\to\infty$, zugleich $m$, $n\to\infty$ lässt so dass

$$\frac{\mu}{m}\to\delta,\quad \frac{\nu}{n}\to\delta$$

wird, so ergibt sich

$$\lim\frac{S_{m,n}-S_{m,\nu}-S_{\mu,n}+S_{\mu,\nu}}{(m-\mu)(n-\nu)}-\underline{\lim}\ S_{\mu,\nu}\geqq -M\,\frac{2(1-\delta)}{\delta}.$$

Da man hier $1-\delta$ beliebig klein auswählen kann, erhält man daraus

$$A-\underline{\lim}\ S_{\mu,\nu}\geqq 0.$$

Wenn man dagegen $\mu>m$, $\nu>n$ setzt, so wird

$$(S_{m,n}+S_{\mu,\nu}-S_{\mu,n}-S_{m,\nu})-(\mu-m)(\nu-n)\,S_{\mu,\nu}$$

$$=-\sum_{i=m+1}^{\mu}\sum_{k=n+1}^{\nu}\Big(\sum_{r=i+1}^{\mu}p_{r,k}+\sum_{t=k+1}^{\nu}q_{\mu,t}\Big),$$

und folglich

$$A-\overline{\lim}\ S_{\mu,\nu}\leqq 0.$$

Damit ist $S_{\mu,\nu}\to A$ bewiesen. Also:

*Satz: Es sei die Doppelreihe*

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}a_{m,n}$$

*summierbar $C(1,1)$ und die Summe sei $A$; ferner sei*

$$mp_{m,n}>-M,\quad nq_{m,n}>-M\quad (\textit{oder}\ (m+n)^2a_{m,n}>-M);$$

*so ist die Reihe konvergent und zwar gleich $A$.*

Dieser Satz ist eine Verallgemeinerung des Hardy-Landauschen Satzes für einfache Reihe. Der spezielle Fall, wo

$$mp_{m,n}=0(1),\quad nq_{m,n}=0(1)$$

ist, wurde schon von Herrn Prof. W. H. Young (1) bewiesen.

**5.** Die Übertragung des Kubotaschen Satzes (2) auf Doppelintegrale ist sehr einfach.

*Satz: Es seien*

$$\varphi(x,y)=\int_0^x\int_0^y\Phi(x,y)dx\,dy$$

$$f(x,y)=\int_0^x\int_0^y\varphi(x,y)dx\,dy,$$

*wo diese Integrale für jede endliche Paare von $(x,y)$ existieren; ferner sei*

$$p(x,y)=\int_0^y\Phi(x,y)dy>-\frac{M}{x},\quad q(x,y)=\int_0^x\Phi(x,y)dx>-\frac{M}{y}\ \textit{für}\ x,y>0,$$

$$(\textit{oder als besonderer Fall}\ (x+y)^2\,\Phi(x,y)>-M).$$

*Dann ergibt sich*

(1) W. H. Young, On multiple Fourier series, Proc. London Math. Soc., (2) 11, 1912-13, p. 167.

(2) Kubota, loc. cit.

$$\varphi(x, y) \to A \quad \textit{für} \quad x, y \to \infty$$

*aus der Annahme*

$$\frac{f(x, y)}{xy} \to A \quad \text{für} \quad x, y \to \infty.$$

Zum Beweise gehen wir von der folgenden Relation aus:

$$f(\xi(1+\varepsilon), \eta(1+\varepsilon)) + f(\xi, \eta) - f(\xi(1+\varepsilon), \eta) - f(\xi, (1+\varepsilon)\eta) = \int_{\xi}^{\xi(1+\varepsilon)} \int_{\eta}^{\eta(1+\varepsilon)} \varphi(x, y) dx\, dy$$

$$= \varepsilon^2 \xi\eta\varphi(\xi, \eta) + \int_{\xi}^{\xi(1+\varepsilon)} \int_{\eta}^{\eta(1+\varepsilon)} (\varphi(x, y) - \varphi(\xi, \eta)) dx\, dy$$

$$= \varepsilon^2 \xi\eta\varphi(\xi, \eta) + \int_{\xi}^{\xi(1+\varepsilon)} \int_{\eta}^{\eta(1+\varepsilon)} \{\varphi(x, y) - \varphi(\xi, y)) + (\varphi(\xi, y) - \varphi(\xi, \eta))\} dx\, dy$$

$$= \varepsilon^2 \xi\eta\varphi(\xi, \eta) + \int_{\xi}^{\xi(1+\varepsilon)} \int_{\eta}^{\eta(1+\varepsilon)} \left\{ \int_{\xi}^{x} p(x, y) dx + \int_{\eta}^{y} q(\xi, y) dy \right\} dx\, dy$$

$$> \varepsilon^2 \xi\eta\varphi(\xi, \eta) - M\varepsilon^2 \xi\eta \log (1+\varepsilon)^2.$$

Daraus folgt für $\xi, \eta \to \infty$

$$A - \overline{\lim}\ \varphi(\xi, \eta) \geqq -M \log (1+\varepsilon)^2,$$

und folglich

$$\overline{\lim}\ \varphi(\xi, \eta) \leqq A.$$

Gleicherweise aus der Betrachtung von

$$f(\xi, \eta) + f(\xi(1-\varepsilon), \eta(1-\varepsilon)) - f(\xi, \eta(1-\varepsilon)) - f(\xi(1-\varepsilon), \eta)$$

$$= \int_{\xi(1-\varepsilon)}^{\xi} \int_{\eta(1-\varepsilon)}^{\eta} \varphi(x, y) dx\, dy$$

erhält man

$$\underline{\lim}\ \varphi(\xi, \eta) \geqq A;$$

daraus erschliesst man

$$\varphi(\xi, \eta) \to A \quad \text{für} \quad \xi, \eta \to \infty,$$

was zu beweisen war.

Sendai, November 1918.

# 不定形 ∞/∞ ニ就テ

## On the Indeterminate Form ∞/∞,

林 鶴 一 (仙臺)

TSURUICHI HAYASHI, Sendai.

1. $\lim_{x\to\infty}\left[\left\{\int_0^x f(x)dx\right\}\Big/x\right]$(1) ガ存在シテ有限値 $\alpha$ ニ等シキトキニ於テ $\lim_{x\to\infty} f(x)$ ガ存在シテ同ジク $\alpha$ ニ等シキ(之レヲ後ニ場合 $A$ トイフ)爲ニハ

$$\lim_{x\to\infty}\left[f(x)-\left\{\int_0^x f(x)dx\right\}\Big/x\right] \tag{1}$$

ガ存在シテ 0 ニ等シキコトガ必要ニシテ且十分ナリ. 即チ

$$\lim_{x\to\infty}\left[\left\{xf(x)-\int_0^x f(x)dx\right\}\Big/x\right]\to 0$$

即チ部分積分法ニヨリテ

$$\lim_{x\to\infty}\left[\left\{\int_0^x xf'(x)dx\right\}\Big/x\right]\to 0 \tag{2}$$

ガ存在シテ 0 ニ等シキコトガ必要ニシテ且十分ナリ. 而シテ此ノ極限ハ L'Hospital ノ法則ニヨリテ分子及ビ分母ヲ各別ニ微分シタルモノニツキ

$$\lim_{x\to\infty} xf'(x)\to 0 \tag{3}$$

ナルトキ存在シテ 0 ニ等シ. 故ニ (3) ハ場合 $A$ ノ爲ニ十分ナル條件ナリ.

更ニ今一度 (2) ニ部分積分法ヲ施セバ

$$\lim_{x\to\infty}\left[\left\{x^2f'(x)-\int_0^x x^2f''(x)dx\right\}\Big/x\right]\to 0 \tag{4}$$

ヲ得. コハ場合 $A$ ノ爲ニ必要ニシテ且十分ナル條件ナリ. 故ニ L'Hospital ノ法則ノ適用ニヨリテ

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(1) 適當ナル注意ノ下ニハ此論文中ノ諸積分ノ下限 0 ヲ正ノ定數ニテ置換フルコトヲ得ルハ勿論ナリ.

$$\lim_{x\to\infty} xf'(x)\to\beta_1 \quad 及 \quad \lim_{x\to\infty}\left[\left\{\int_0^x x^2f''(x)dx\right\}\Big/x\right]\to\beta_1 \tag{5}$$

ハ場合 $A$ ノ爲ニ十分ナル條件ナリ. 而シテ (5) ノ第二ニ於テ分母子ヲ各別ニ微分セシモノヲ作レバ

$$\lim_{x\to\infty} xf'(x)=\lim_{x\to\infty} x^2f''(x)\to\beta_1 \tag{6}$$

ガ場合 $A$ ノ爲ニ十分ナル條件ナリ.

更ニ (4) ニ今一度部分積分法ヲ施セバ

$$\lim_{x\to\infty}\left[\left\{x^2f'(x)-x^3f''(x)/3+\int_0^x x^3f'''(x)dx/3\right\}\Big/x\right]\to 0$$

ガ場合 $A$ ノ爲ニ必要ニシテ且十分ナル條件ナリ. (4) ヨリ (6) ニ達スルト同樣ニセバ

$$\lim_{x\to\infty}\left[xf'(x)-x^2f''(x)/3\right]=-\frac{1}{3}\lim_{x\to\infty} x^3f'''(x)\to\beta_2$$

ガ場合 $A$ ノ十分ナル條件ナリ.

更ニ一般ニ同樣ノ考ヲ使用スレバ

$$\begin{aligned}\lim_{x\to\infty}\Big[xf'(x)&-\frac{1}{3}x^2f''(x)+\frac{1}{3\cdot 4}x^3f'''(x)-\cdots\cdots\\&+(-1)^n\frac{2}{n!}x^{n-1}f^{(n-1)}(x)\Big]\\&=(-1)^n\frac{2}{n!}\lim_{x\to\infty} x^nf^{(n)}(x)\\&\to\beta_{n-1}\end{aligned}$$

ガ場合 $A$ ノ十分ナル條件ナリ.

偖テ (3) ニ於テ

$$x=G(y),\quad \int_0^x f(x)dx=F(y)$$

ト置キテ多少ノ條件 (1) ヲ加ヘテ進ムトキハ

$$\lim_{y\to\infty}\frac{F(y)}{G(y)}\to\alpha \quad 且 \quad \lim_{y\to\infty}\frac{G(y)}{G'(y)}\cdot\frac{d}{dy}\frac{F'(y)}{G'(y)}\to 0$$

(1) タトヘバ $G(y)$ ガ $y$ ノ增加スルトキ單調ニ增加ストイフガ如キ.

ナルトキハ

$$\lim_{y\to\infty}\frac{F'(y)}{G'(y)}\to\alpha$$

ナルコトヲ證明シ得.

同様ニ (6) ニヨリテハ

$$\lim_{y\to\infty}\frac{F(y)}{G(y)}\to\alpha$$

且

$$\lim_{y\to\infty}\frac{G(y)}{G'(y)}\cdot\frac{d}{dy}\frac{F'(y)}{G'(y)}=\lim_{y\to\infty}\frac{\{G(y)\}^2}{G'(y)}\cdot\frac{d}{dy}\left\{\frac{1}{G'(y)}\cdot\frac{d}{dy}\frac{F'(y)}{G'(y)}\right\}\to\beta_1$$

ナルトキ

$$\lim_{y\to\infty}\frac{F'(y)}{G'(y)}\to\alpha$$

ナルコトニ到達ス.

但シ條件 (3) 及ビ (6) ハ窪田君ノ論文 ([1]) 中ノ定理 5 ニ含マルルモノナルガ, 其ノ證明ノ簡單ナル點ニ於テ採ルベキモノアルベシ.

**2.** 次ギニ

$$\lim_{x\to\infty}\left[\int_0^x g(x)\left\{\int_0^x f(x)dx\right\}dx\Big/\int_0^x g(x)dx\right] \tag{7}$$

ガ存在シテ $\alpha$ ニ等シキトキ

$$\lim_{x\to\infty}\int_0^x f(x)dx$$

ガ亦存在シテ $\alpha$ ニ等シキ(之ヲ場合 $B$ トイフ)爲ニハ前ト同様ニシテ

$$\lim_{x\to\infty}\left[\int_0^x f(x)dx-\frac{\int_0^x g(x)\left\{\int_0^x f(x)dx\right\}dx}{\int_0^x g(x)dx}\right]$$

ガ存在シテ 0 ニ等シキコトガ必要ニシテ且十分ナリ. 即チ

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([1]) 本誌第 15 卷, 頁 314–322 ニ於ケル論文 *Einige Sätze den Grenzwert betreffend* 參照.

$$\lim_{x\to\infty}\left[\left\{\int_0^x f(x)dx.\int_0^x g(x)dx-\int_0^x g(x)\left\{\int_0^x f(x)dx\right\}dx\right\}\Big/\int_0^x g(x)dx\right]\to 0$$

即チ分子ノ第二項ニ部分積分法ヲ施シテ

$$\lim_{x\to\infty}\left[\int_0^x f(x)\left\{\int_0^x g(x)dx\right\}dx\Big/\int_0^x g(x)dx\right]\to 0 \tag{8}$$

ガ場合 $B$ ノ爲ニ必要ニシテ且十分ナル條件ナリ.

再ビ假定ニ立チ戻リテ考フレバ

$$\lim_{x\to\infty}\left[\int_0^x g\left\{\int_0^x f\,dx\right\}dx\Big/\int_0^x g\,dx\right]\to\alpha$$

且

$$\lim_{x\to\infty}\left[\int_0^x f\left\{\int_0^x g\,dx\right\}dx\Big/\int_0^x g\,dx\right]\to 0$$

ナルトキニハ

$$\lim_{x\to\infty}\int_0^x f(x)dx\to\alpha$$

ナリトイフヲ得. 此ノ兩條件ノ分子ニ於テハ $f$ ト $g$ トガ交換サレタルノミノ差ガ起レルハ注意スベキ面白キコトナリ. 但シコレモ亦窪田君ノ論文中ノ定理 1 ノ證明中ニ導入セラレタルモノナリ.

若シ $g(x)\equiv 1$ トスレバ

$$\lim_{x\to\infty}\left[\left\{\int_0^x\int_0^x f(x)dx\,dx\right\}\Big/x\right]\to\alpha$$

ナルトキ

$$\lim_{x\to\infty}\int_0^x f(x)dx\to\alpha$$

ナルガ爲ニ必要ニシテ且十分ナル條件ハ (8) ニヨリテ

$$\lim_{x\to\infty}\left[\left\{\int_0^x xf(x)dx\right\}\Big/x\right]\to 0$$

ナリ. コレハ無限級數ニ關スル Tauber ノ定理 (1) ニ相應スルモノナリ.

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(1) A. Tauber: *Ein Satz aus der Theorie der unendlichen Reihen*, Monatshefte für Mathematik und Physik, 8. Jahrgang (1897), p. 273.

今又 $\int_0^x f(x)dx=\varphi(x)$ ト置ケバ, コハ

$$\lim_{x\to\infty}\left[\left\{\int_0^x \varphi(x)dx\right\}\Big/x\right]\to\alpha \quad 且 \quad \lim_{x\to\infty}\varphi(x)\to\alpha$$

ナル爲ニ必要ニシテ且十分ナル條件ハ

$$\lim_{x\to\infty}\left[\left\{\int_0^x x\varphi'(x)dx\right\}\Big/x\right]\to 0$$

ナルコトヽナリ, コハ前節ニ得タルトコロノ條件 (2) トナル.

又 (8) ハ L'Hospital ノ法則ニヨリ其ノ分母子ヲ各別ニ微分シテ

$$\lim_{x\to\infty}\left[\left\{f(x)\int_0^x g(x)dx\right\}\Big/g(x)\right]\to 0 \tag{9}$$

トナスヲ得. 是レ場合 $B$ ノ爲ニ十分ナル條件ナリ. 是ニ於テ $g(x)\equiv 1$ トスレバ

$$\lim_{x\to\infty}\left[\left\{\int_0^x\int_0^x f(x)dx\,dx\right\}\Big/x\right]\to\alpha$$

且

$$\lim_{x\to\infty} xf(x)\to 0$$

ナルトキハ

$$\lim_{x\to\infty}\int_0^x f(x)dx\to\alpha$$

ナリ.

又 $h(x)\Big/\int_0^x g(x)dx$ ガ有界的 (bounded) ナルガ如キ函數 $h(x)$ ノ存在スルトキニハ

$$\lim_{x\to\infty}\frac{f(x)\int_0^x g(x)dx}{g(x)}\cdot\frac{h(x)}{\int_0^x g(x)dx}\to 0$$

ナルトキ (9) ハ成立ス. 即チ

$$\lim_{x\to\infty} f(x)\,h(x)/g(x)\to 0$$

ナルトキ (9) ハ成立ス. 故ニ $h(x)\Big/\int_0^x g(x)dx$ ガ有界的ナルトキ場合 $B$ ノ爲ニハ

$$\lim_{x\to\infty} f(x)\cdot h(x)/g(x)\to 0 \tag{10}$$

ナレバ十分ナリ.

$h(x)\equiv x$ トスレバコハ窪田君ノ定理 1 ニ對應スル定理ナリ. 其上ニ特ニ $g(x)\equiv 1$ トスレバ

$$\lim_{x\to\infty}\left[\left\{\int_0^x\int_0^x f(x)dx\,dx\right\}\Big/x\right]\to a$$

且

$$\lim_{x\to\infty} xf(x)\to 0$$

ナルトキハ

$$\lim_{x\to\infty}\int_0^x f(x)\,dx\to a \tag{11}$$

ナリ. 之レ窪田君ノ定理 2 ニ對應ス.

此等ノ定理ヨリモ亦 $\lim\limits_{y\to\infty} F(y)/G(y)\to a$ ナルトキ

$$\lim_{y\to\infty} F'(y)/G'(y)\to a$$

ナリト斷定スルニ要スル十分條件ヲ作製スルコトヲ得ベシ.

**3.** 窪田君ノ定理 1 ノ一ツノ條件即チ

$$\int_0^\infty f(x)^2\left\{\int_0^x g(x)dx\right\}dx$$

ガ收斂ナリトノコトヲ一層緩ムルニ, 此ノ積分ハ

$$f(x)^2\int_0^x g(x)dx < \frac{\text{const.}}{x^{1+2s}},\quad s>0,$$

ナルトキ收斂スルニヨリ

$$xf(x)^2\int_0^x g(x)dx < \frac{\text{const.}}{x^{2s}},\quad s>0$$

ヲ以テ同定理ノ十分條件トナスヲ得. 然ルニ同定理ノ條件トシテ

$$x\Big/\int_0^x g(x)dx < \text{const.}$$

ナルコトアルヲ以テ

$$x^2 f(x)^2 < \frac{\text{const.}}{x^{2s}}, \quad s > 0$$

ヲ得. 故ニ

$$\left| xf(x) \right| < \frac{\text{const.}}{x^s}, \quad s > 0$$

ナルコトヲ以テ窪田君ノ定理 1 ノ一ツノ條件ヲ置換シ, ヨリテ以テ一ツノ定理トスルコトヲ得ベシ. 此形ニ於ケル定理ト前節ニ於ケル定理 (10) 及ビ (11) トハ頗ル相類似セルヲ見ル.

大正七年十一月.

# 抄　錄　短　評

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**Annales Scientifiques de l'École Normale Supérieure**, (3), t. 35, No. 3, 4, Mars-Mai, 1918.

J. Boussinesq, Complément à un récent mémoire des Annales scientifiques de l'École Normale Superieure, sur la poussée des terres et l'état ébouleux, avec quelques idées générales sur la mécanique des semi-fluides et application de ces idées aux corps plastiques.

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R. Alezais, Sur le système de $n$ équations du second degré. R. Bricard, Sur une propriéte caractéristique des coniques homofocales. F. Balitrand, Relations entre les rayons de courbure de deux courbes affines. E. Jablonski, Sur la distribution des nombres premiers absolus. M. Weill, Propriété des coniques et des quadriques à centres. G. Fontené, Sur les cercles de Pappus: Formule de Schubert généralisée. F. Balitrand, Sur la condition pour que les tangentes aux pieds des normales issues d'un point à une ellipse touchent un cercle. Ph. du Plessis, Remarques géométriques sur la question de concours de l'École Polytechnique en 1918.

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G. Giraud, Sur le attachement à la théorie des fonctions hyperabéliannes d'une certaine équation aux dérivées partielles du second ordre, avec généralisation à un nombre quelconque de variables. A. Angalesco, Sur l'approximation simultanée de plusieurs intégrales définies. Riquier, Sur une propriété des fonctions d'une nombre quelconque de variables imaginaires. E. Gau, Sur les caractéristiques des équations aux dérivées partielles du second ordre. J. Drach, Sur les groupes complexes de rationalité et sur l'integration par quadratures. T. Lalesco, Sur les fonctions polygonales périodiques. R. Garnier, Solution élémentaire du problème de l'inversion des fonctions elliptiques. R. Goormaghtigh, Généralisations des théorèmes de Jamet sur la courbure des courbes triangulaires, des courbes et des surfaces tétraédrales symétriques.

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H. Vilat, Quelques récents progrès des théories hydrodynamiques. H. Vergne,

Théorie élémentaire du mouvement de précession et de la dérivation des projectiles. M. Fréchet, Sur la notion de voisinage dans les ensembles abstraits. C. de Waard, Un écrit de Beaugrand sur la méthode des tangentes de Fermat à propos de celle de Descartes. F. Gonseth, Un théorème relatif a deux ellipsoïdes confocaux. F. Gonseth, Un théorème relatif a deux ellipsoïdes confocaux (suite et fin). Lalesco, Sur l'addition des noyaux non orthogonaux. A. Buhl, Sur les sommes abéliennes de volumes cyclido-coniques. G. Julia, Sur les valeurs limites de l'intégrale de Poisson relative à la sphère en un point de discontinuité des données. M. Bottaso, Quelques remarques sur le système vectoriel de M. Burali-Forti et Marcolongo. M. Petrovitch, L'aire des surfaces de révolution.

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# 雜　錄　彙　報

## 歐米諸大學ノ課程

### 伊太利

#### ぼろーにぁ大學 (1918-19)

ぶるがっち Burgatti, 天體力學 (3). どなーち Donati, 電氣力學, 電氣光學, 相對律 (3), えんりけす Enriques, 數學ノ基礎 (i), 幾何學, 極微解析, 力學, 古代希臘ニ於ケル宇宙論; (ii), 諸原理ノ近世ノ批判 (3). ぴんけるれ Pincherle, 「ヴォルテラ」及ビ「フレシェー」ノ functional calculus, 微分方程式ニ於ケル解存在ノ定理, 積分方程式 (3).

#### かたにあ大學 (1918-19)

ちぽら Cipolla, 「フーリェ」ノ級數, 球函數及ビ圓柱函數, 複素變數ノ函數論, 楕圓函數, 應用 (4). だにえーれ Daniele, 數理物理學ニ於ケル微分方程式及ビ應用 (4). すこるざ Scorza, 高次空間ノ幾何學及ビ應用 (4). せゔぇりーに Severini, 變分學 ($4\frac{1}{2}$).

#### ぜのあ大學 (1918-19)

ろーりあ Loria, 微分幾何學 (3). てどーね Tedone, 二元偏微分方程式及ビ其物理學上ノ應用 (3). 擔任者未定, 解析學 (3).

#### めっしな大學 (1918-19)

ぼったそ Bottasso, 「ヴェクトル」解析, 「ニウトン」ノ「ポテンシァル」, 彈性論 (3). からぷそ Calapso, 楕圓函數論 (3). じゃんべり Giambelli, 高次空間ノ解析幾何學, 代數的消去ノ幾何學的理論 (3).

### なぽり大學 (1918-19)

あもでお Amodeo, 數學史(上古ヨリ紀元千二百年代迄) (3). でるれ Del Re, 「グラスマン」ノ $n$ 次元解析, 微分幾何學及ビ力學ヘノ應用 (4½). まるころんご Marcolongo, 「フーリエ」級數ノ應用 (3). もんてざーの Montesano, 空間ノ birational 變換論, 對合的 birational 變換 (3). ぱすかる Pascal, 解析學特選題目 (3). びんと Pinto, 幾何光學 (3).

### ぱどぅあ大學 (1918-19)

だるかいす D'Arcais, 複素變數ノ函數論, 積分方程式 (4) がっつぁにーが Gazzaniga, 數論 (3). れゔぃーちゔぃた Levi-Civita, 電磁場 (4). りっち Ricci, 絕對微分學及ビ應用 (4). せゔぇーり Severi, 微分幾何學 (4).

### ばれーるも大學 (1918-19)

ばにぇーら Bagnera, 一般解析函數論, 整函數, 線狀微分方程式 (3). ふらんきす Franchis, 代數曲面ノ幾何學 (3). じぇびあ Gebbia, 連續體ノ力學, 「ニウトン」ノ「ポテンシアル」及ビ對數「ポテンシァル」, 流體靜力學及ビ流體動力學 (4½). しにょりーに Signorini, 理論重學特選題目, 特ニ彈性論 (3).

### ばうぃあ大學 (1918-19)

べるぞらーり Berzorari, 代數曲線及ビ代數曲面 (3). ちぞっち Cissoti, 流體力學 (3). ぢぇるばるぢ Gerbaldi, 複素變數ノ函數論及ビ楕圓函數 (3). ゔぃゔぁんち Vivanti, 實變數ノ函數論 (3).

### びざ大學 (1918-19)

べるちーに Bertini, 代數曲線ノ幾何學 (3). びあんき Bianchi, 變換ノ連續群ノ理論 (4½). でぃーに Dini, 「フーリェ」級數及ビ所設ノ區間ニ於ケル實變數函數ノ解析的表示ニ關スル更ニ一般ナル展開 (4½). まっぢ Maggi, 不易及ビ變易電磁場 (4½). 擔任者未定, 理論重學 (3)

### ろーま大學 (1918-19)

びすこんちに Bisconcini, 應用解析幾何 (3). ぼんぴあに Bompiani, 空間ノ接觸變換及其連續群 (3). かすてるぬおーぼ Castelnuovo, 代數方程式ト置換群 (3). くるでり Crudeli, 連續變換群ノ理論 (3). しら Silla, 力學ノ微分方程式 (3). ぼるてら Volterra, 波動論 (3), 流體力學 (3).

### とりの大學 (1918-19)

ぼっじお Boggio, 廻轉セル液體ノ平衡形態 (3). ふびに Fubini, もぢゅらー函數, おーともるふ函數, ふっくすノ函數, 有理函數ノ常微分方程式 (3). せぐれ Segre, 代數的「ラインコップレックス」(3). そみりあな Somigliana 熱力學, 氣體論, 熱傳導 (3)

## 瑞 西

### ばーぜる大學 (1918-19 冬學期)

まちぃす W. Matthies, 數理物理學せみなりー (2). 擔任者未定, 解析幾何學 (4), 高等數學特選題目 (4), 演習 (1), 數學せみなりー (1). すぴーす O. Spiess, 微分積分學 (4), 微分方

程式(特ニ物理學ニ關スルモノ) (3), ぷろせみなりー (1), せみなりー (1). ふらっと R. Flatt, 教育せみなりー, 數理自然科學 (3), 射影幾何學 (2). くなっぷ M. Knapp, 歲差運動及攝動, 球面星學 (2), 通俗天文學 (1). 「とれみー」ノ Almagest ニ關スル講演 (1), 天文臺實習 (5).

## べるん大學 (1918-19 冬學期)

ふーばー G. Huber, 平面曲線論 (3), 楕圓函數及てーた函數 (2), せみなりー (1). くれりぇ Crelier, 積分學 (2), 代數解析 (2), 解析幾何學 (2), 畫法幾何學 (2), 非ゆーくりっど幾何學 (2). まうでるり Mauderli, 天體觀測實習 (2). ういす Wiss, 天文學ニ必要ナル計算, 天文學上ノ表ノ利用及作製 (2), 時辰ノ基礎原理, 復活日ノ計算, 曆 (1), 天文講演. べるりーなー Berliner, 高等代數學 (2), 解析整數論 (1), 整數論 (1). ふーばー Huber, 綜合幾何學 (1), 微分幾何(空間曲線及曲面) (1). もーざー Moser, 老廢者保險ノ基礎 (2), 保險學特選題目, 數理保險學せみなりー (2). ぼーれん Bohren 傷害保險ノ實地及瑞西國傷害保險制度 (2), 最小自乘法 (1).

## ふらぃぶるぐ大學 (1918-19 冬學期)

ぷらんしゅれる Plancherel, 解析幾何學 (3), 微分方程式 (3), 解析幾何學演習 (1), せみなりー (2). だにえるす Daniëls, 微分方程式 (4), 微分方程式演習 (1), 熱力學 (3), 公算論 (2), ぐらすまんノ廣延論 (1).

## ぜねうぁ大學 (1918-19 冬學期)

けーらー Cailler, 微分積分學 (3), 演習 (2), 理論重學 (3), 演習 (2), 微分積分學補充 (1), 數學解析講義, 積分方程式 (2). ふぇーる Fehr, 高等數學初步 (3, 代數學及解析幾何學講義 (1), 高等數學初步ニ關スル演習 (2), 射影幾何學 (1), 高等數學せみなりー(代數學ヨリ選バレタル題目) (2), 初等數學せみなりー(數學方法論) (1). ごーちぇ Gauthier, 物理學的天文學, 其他私講師ノ講義トシテ次ノモノアリ. みりまのふ Mirimanoff, 公算論 (2). ちーあしー Tiercy, 物理的重學, 變形論 (1).

## ろーざんぬ大學 (1918-19 冬學期)

あむすたいん Amstein, 函數論 (3), 積分學補充 (2). ぢゅーま G. Dumas, 微分積分學 (5), 演習 (2), 解析學問題 (2), せみなりー (1). らこむべ Lacombe, 畫法幾何學 (4), 製圖 (4), 解析幾何 (2), 位置幾何學及演習 (3). めーやー Mayor, 理論重學 (4), 演習 (1), 數理物理學 (2). めーらーる Maillard, 極微解析及其ノ科學上ノ應用 (4), 球面星學 (3) 理論重學 (2). ぢゅーま S. Dumas, 公算論 (3). 其他私講師ノ講義トシテ次ノモノアリ. ぢぁこってっと Ch Jaccottet, 實變數函數論 (2). ぱしぅど Paschoud 數理物理學緒論 (2).

## ちゅーりひ大學 (1918-19 冬學期)

ふゅーたー, Fueter, 自然科學ノ數學的取扱ヒ (3), 演習 (1), おーともるふ函數論 (4), すばぃざー 教授ト共同ノせみなりー (1). すばぃざー Speiser, 微分積分學 (4), 代數方程式ニ關スルがろあノ理論 (3), 演習 (1). うぉるふぁー Wolfer, 天文學緒論 (3) 演習 (2), 遊星及彗星ノ軌道ノ決定法 (2). 其他私講師ノ講義トシテ次ノモノアリ. ごんせーと Gonseth, 應用數學 (4). べるないす Bernays, 集合論 (3).

ちゅーりひ市立工業學校 (1918-19 冬學期)

ひるしゅ Hirsch, 高等數學 (6), 演習 (2). ふらねる Franel, 高等數學 (6), 演習 (2). ぐろすまん Grossmann 畫法幾何學 (4), 演習 (4), 射影幾何學 (4). わいる Weyl, 解析幾何學 (2), 演習 (1). こるろす Kollros, 畫法幾何學 (4), 演習 (4). まいすなー Meissner, 重學 (4), 演習 (2). ふるゐっつ Hurwitz, 楕圓函數論 (4), 高等整數論 (2). ふるゐっつトこるろす Hurwitz u. Kollros, せみなりー (2). わいる Weyl, 電磁場論 (4), 積分方程式 (2). まいすなー Meissner, 振動論及波動論 (2). べしゅりん Baeschlin 測量學, 高等測地學 (3). うぁるふぁー Wolfer, 天文學結論 (3), 演習 (2), 遊星及彗星ノ軌道決定法 (2). あむべるぐ Amberg 恩給保險數學 (2). ぶらんでんべるがあ Brandenberger, 數學自然科學教育緒論 (2). ぽりあ Pólya, 實數解析論 (2). きーなすと Kienast, 解析重學 (2), 隨意科トシテ. あむべるぐ Amberg, 恩給保險數學 (2). ばいえる Beyel, 計算尺, 演習 (1) 畫法幾何學 (2), 圓錐曲線 (1). ぶれんたの Brentano, 光學及電氣學ノ範圍ニ於ケルえれくとろんノ理論 (2). ごんせーと Gonseth, 應用數學特選題目 (2), 三次曲面 (1). けるらー J. Keller, 畫法幾何學特選題目 (2). くらふと Kraft, 世界ノ根源力 (1), 幾何學的解析 (3). 運動學 (3). ぽりあ Pólya 函數論ノ幾何學的應用.

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## A remark on a paper of Prof. Kakeya in Vol. 2 of this Journal.

Prof. Kakeya, in his paper, *On the limits of the roots of an algebraic equation with positive coefficients*, proved the following theorem:

The absolute values of the roots of an equation with real positive coefficients

$$p_n x^n + p_{n-1} x^{n-1} + p_{n-2} x^{n-2} + \cdots + p_1 x + p_0 = 0$$

lie between the greatest and the least of $\frac{p_0}{p_1}, \frac{p_1}{p_2}, \cdots, \frac{p_{n-1}}{p_n}$.

Recently, I happened to find the following question proposed by G. Eneström in L'Intermédiaire des Mathématiciens, Vol. 2 (1895), p. 418, to which no answer is yet given.

Dans une recherche sur la théorie des caisses de retraites, j'ai été, il y a quelques années, amené au théorème suivant: Soit $x^n + a_1 x^{n-1} + a_2 x^{n-2} + \cdots + a_{n-1} x + a_n = 0$ une équation où $1 > a_1 > a_2 > \cdots > a_{n-1} > a_n > 0$; appelons, de plus, $\alpha$ la plus petite et $\beta$ la plus grande des fractions $\frac{a_1}{1}, \frac{a_2}{a_1}, \frac{a_3}{a_2}, \cdots, \frac{a_n}{a_{n-1}}$; alors les valeurs absolues des toutes racines de l'équation sont comprises entre $\alpha$ et $\beta$. Ce théorème a t il été indiqué dans quelque Traité sur théorie des équations? (K. Y.)

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## Corrigenda in Mr. M. Lecat's paper, this Journal, vol. 5, 1914.

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| Page | 119, | line | 6, | instead | of | $E\left[\frac{n}{1}\right]$ | read | $E\left[\frac{n}{2}\right]$, |
| ,, | ,, | ,, | ,, | ,, | | parmanent | ,, | permanent, |
| ,, | 120, | ,, | 17, | ,, | | $\bar{\theta}$ and | ,, | $\sigma$ and, |
| ,, | 122, | ,, | 13, | ,, | | $y+g$ | ,, | $y+9$, |
| ,, | ,, | ,, | 15, | ,, | | $\alpha$ | ,, | $\sigma$, |
| ,, | 123, | ,, | 16, | ,, | | $(y+1)^2$ | ,, | $(y-1)^2$, |
| ,, | ,, | ,, | 21, | ,, | | $\gamma_g$ | ,, | $\gamma_k$ |
| ,, | 124, | ,, | 5, | before | | $g$ | insert | when |
| ,, | 125, | ,, | last, | ,, | | $\nu_N-1$ | ,, | $\nu_{N-1}$, |
| ,, | 126, | ,, | 33, | ,, | | actinoited | ,, | actinoid, |
| ,, | 129, | ,, | 28, | ,, | | no | ,, | but, |
| ,, | 130, | ,, | 17, | ,, | | $\nu-1$ | ,, | $\nu=1$, |
| ,, | ,, | ,, | 22, | ,, | | indicative | ,, | inactive, |
| ,, | 131, | ,, | 23, | ,, | | $n\times$ | ,, | $n+$, |
| ,, | 133, | ,, | 28, | ,, | | nor | ,, | or, |

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## Mathematische Zeitschrift.

講和會議ノ進捗ハ向後如何ニ變ハリ行カンカ殆ド豫測シ難キモノアリ. 此ノ大戰爭ニ於テ我等聯合國共同ノ敵トシテ困憊ノ極ニ達スルマデ奮鬪シタル獨逸ガ戰前學術ノ淵藪タルノ觀アリシハ諸君ノ知ラルヽトコロナルガ講和後モ亦文化的競爭ニ於テ他國ノ後ニ附クガ如キコト恐ラクハ之レ無カルベシ. 戰爭遂行ノ最中 1918 年ノ初頭ニ於テ有力ナル數學專門ノ雜誌 Mathematische Zeitschrift ノ創刊セラレタルガ如キハ如何ニ純理學問ニ熱中スル國民タルカヲ知ルニ足ル. 同誌ハ同年間二卷ヲ發行セリ. 第一卷ハ 428 頁, 第二卷ハ 474 頁ニシテ皆四號 Heft ヨリ成ル, 形ハ Mathematische Annalen ニ類シ, 主編輯者ハ伯林 L. Lichtenstein, 之ヲ援クル者ハ K. Knopp, E. Schmidt 及 I Schur ニシテ, 尙ホ顧問トシテ Blaschke, Fejér, Herglotz, Kneser, Landau, Perron, F. Schur, Study 及ビ Weyl ノ名ヲ列ス. 發行所ハ Crelle's Journal ト同樣ニ伯林 Julius Springer ニシテ各卷 24 馬克ノ價ナリ. 每年二卷ヲ發行スルノ定メナルガ, 本年ノ分ハ一號モ未ダ到着セズ, 數學ノ進步ヲ知リテ其ノ發達ノ爲ニ貢獻セントスルモノニハ必讀スベキ內容ノ一大新雜誌ナリ. (T. H.)

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## 諸學者ノ消息

英吉利けむぶりっぢ Trinity College 及倫敦 University College ノ教授わっとそん G. N. Watson 氏ハひーす Heath 教授ノ後任トシテばーみんがむ大學數學科教授トナレリ.

北米合衆國ぺんしるゔぁにあ大學ノくらいん氏 Dr. J. R. Kline ハいぇーる大學ニ於ケル Schefield Scientific School ノ數學教師トナレリ.

伊太利ぴざ大學教授ぴぜっぢ氏 Paolo Pizzetti ハ 1916 年 4 月 16 日 58 歲ニテ逝去セリ.

北米合衆國ぺんしるゔぁにあ大學教授いゔぁんす氏 H. B. Evans ハ Towne Scientific School ノ長トナレリ,

英吉利倫敦 City and Guilds Technical College ノ教授へんりっし氏 O. Henrici (菊池森爾氏共譯ノ合同圖形論ノ原著者) ハ 1918年 8 月 10 日 78 歳ニテ逝去セリ.

北米合衆國かーねる大學ノしるゔぁまん氏 Dr. L. L. Silverman ハだーとます College ニ於ケル數學教授トナレリ.

北米合衆國ぺんしるゔぁにあ大學ノびーる氏 Dr. F. W. Beal ハわしんとん大學ノ數學科助教授トナレリ.

北米合衆國わしんとん大學ノうゑーあ氏 Dr. L. E. Wear ハするーぷ College ノ教授トナレリ.

大正 8 年 (1919) 1 月 8 日廣島高等師範學校講師波木井九十郎氏逝去ノ報突如トシテ來タル, 同氏ハ嘗テ仙臺第二高等學校教授トシテ令聞アリ. 其ノ教授セラレタル門弟ニシテ要路ニ立テルモノ多シ. 更ニ仙臺第二高等學校內ニ第三臨時教員養成所ノ設ケラルヽヤ其主事トナリ, 同養成所ガ東北帝國大學總長ノ管理スル所トナリ終ニ廢止セラルヽマデ實ニ同所ノ主腦タリシ, 其間養成セラレタル中等學校數學科教員ハ約二百ニ垂ントス, 其ノ效績至大ナリトイフベシ. 又傍ラ文部省視學委員トシテ全國ニ涉タリテ視察ヲ遂ゲ數學教育上ニ盡力スルコト尠少ニアラズ, 近時又日本中等教育數學會ノ創立ニ際シテモ力ヲ添ヘラレ, 向後ノ發展ノ同氏ニ期待セラルルモノ多カリシニ夭折セラレタルハ誠ニ遺憾ナリ. (T. H.)

大正 8 年 (1919) 2 月 17 日京都帝國大學理科大學教授西內貞吉氏並ニ同助教授園正造氏ハ理學博士ノ學位ヲ授與セラル. 前者ハ同大學總長ノ推薦ニヨリ, 後者ハ相等ノ基礎的概念竝環及等率ノ性質ト題スル論文ノ審査ヲ經タリ.

大正 8 年 (1919) 2 月 19 日 靜岡縣富士郡元吉原村鈴川驛ニ退隱セラレタル川北敦鄰翁危篤ノ報傳ハリ超ヘテ同 22 日死去ノ報達セリ. 同翁ガ我數學界ニ對シテ盡セル努力ハ實ニ大ナルモノアリ. 著譯書ヲ以テ明治初年ノ我學界ヲ助ケタルノ外, 日本數學物理學會ノ前身タル東京數學會社ノ創立ニ向ツテハ有效ノ熱誠ヲ以テ盡サレタリ. 晩年ニ及ビテ陸軍陸地測量部ニ入リテ三角測量ノ事ニ從ハレシガ, 終ニ鄉里鈴川ニ退キテ老軀ヲ養ハレ居タリシナリ. 同氏ハ和算ヲ御粥安本, 內田恭等ニ學ビテ其ノ堂ニ入リ, 修學ノ筆記ハ積ンデ百卷ニ及ビ數理起源ト稱ス. 和算的問題及ビ其解式ヲ蒐集セルモノナルガ故ニ頗ル重寶ナリ. 其他此方面ノ著述モ尠カラズ. 關流正統第七傳ノ稱ヲ以テ一方ノ霸タリ. 其ノ著述等ニ就キテハ更ニ詳細ニ研究シテ傳記ト共ニ記載スルノ日アルベシ. (T. H.)

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# 東北數學雜誌

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# THE TÔHOKU MATHEMATICAL JOURNAL

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**T. Hayashi**

with the cooperation of

*M. Fujiwara, J. Ishiwara,*
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